

LOCAL ENERGY DECAY FOR SCALAR FIELDS ON TIME DEPENDENT NON-TRAPPING BACKGROUNDS

JASON METCALFE, JACOB STERBENZ, AND DANIEL TATARU

ABSTRACT. We consider local energy decay estimates for solutions to scalar wave equations on nontrapping asymptotically flat space-times. Our goals are two-fold. First we consider the stationary case, where we can provide a full spectral characterization of local energy decay bounds; this characterization simplifies in the stationary symmetric case. Then we consider the almost stationary, almost symmetric case. There we establish two main results: The first is a “two point” local energy decay estimate which is valid for a general class of (non-symmetric) almost stationary wave equations which satisfy a certain nonresonance property at zero frequency. The second result, which also requires the almost symmetry condition, is to establish an exponential trichotomy in the energy space via finite dimensional time dependent stable and unstable sub-spaces, with an infinite dimensional complement on which solutions disperse via the usual local energy decay estimate.

CONTENTS

1. Introduction	2
1.1. The geometric setup	2
1.2. Local energy norms	4
1.3. Energy estimates	5
2. Main results	5
2.1. The stationary case.	5
2.2. The nonstationary case	10
2.3. An overview of the paper	11
3. Exterior estimates for asymptotically flat perturbations of Minkowski	11
4. High frequency analysis	14
5. Carleman estimates and the medium frequency analysis	17
5.1. Carleman estimates for wave equations	17
5.2. The medium frequency estimate	20
6. Low frequency analysis	21
7. Two point local energy decay and energy growth	24
7.1. An extension argument	24
7.2. Time frequency localization	25
7.3. The exponential energy dichotomy	27
8. Spectral theory in the time independent case	28
8.1. Local energy decay vs. resolvent bounds	28
8.2. Real resonances and the limiting absorption principle	29
8.3. Nonzero resonances and LE resolvent bounds	31
8.4. Stability in LE resolvent bounds	31
8.5. High frequency analysis.	32
8.6. The general case	32
9. Spectral Theory and Exponential Trichotomies	32
9.1. Spectral theory in the time independent case	32
9.2. Perturbation theory of S^\pm	35
9.3. Global construction of $P^{\pm,0}$ via Perron sums	36

The first author is supported in part by NSF grant DMS-1054289, and the third author was supported by NSF grant DMS-1266182 as well as by a Simons Investigator Award from the Simons Foundation.

1. INTRODUCTION

The aim of this paper is to contribute to the understanding of local energy decay bounds for the wave equation on non-trapping, asymptotically flat space-times. This is a well understood question for small perturbations of the Minkowski space-time. Our aim here, instead, is to consider large perturbations. In that, our goals are two-fold.

First we consider the *stationary* case. There we provide a full spectral characterization of the local energy decay estimates in terms of the eigenvalues and resonances of the corresponding elliptic problem. There are three such objects which are of interest to us:

- Complex eigenvalues outside the continuous spectrum \mathbb{R} and in the lower half-space,
- Zero eigenvalues/resonances, and
- Nonzero resonances embedded inside the continuous spectrum.

Our main result here asserts that local energy decay holds iff none of these three obstructions occurs.

One significant simplification occurs in the *symmetric* case, where no nonzero resonances can occur inside the continuous spectrum. There our results are consistent with the standard spectral theory for self-adjoint elliptic operators. In that case, we also consider the problem of continuity of our spectral assumptions along one parameter families of operators; the main idea being that complex eigenvalues can only emerge via the zero mode. If complex eigenvalues do occur, a slightly more complicated picture emerges, and the flow splits into two finite dimensional subspaces where exponential growth, respectively decay occurs, and a bulk part with uniform energy bounds.

Secondly, we study the case of *time dependent* operators and show that the results in the stationary case are stable with respect to perturbations. More precisely, we consider almost symmetric, almost stationary operators which satisfy a quantitative zero spectral assumption uniformly in time, but allowing for eigenvalues off the real axis. Then we establish an exponential trichotomy, splitting the energy space as a direct sum of three subspaces as follows:

- A finite dimensional subspace of spatially localized, exponentially increasing solutions, associated to eigenvalues in the lower half-space
- A finite dimensional subspace of spatially localized, exponentially decreasing solutions, associated to eigenvalues in the upper half-space
- A remaining infinite dimensional subspace of bounded energy solutions with good local energy bounds.

If there are no eigenvalues off the real axis then only the last subspace is nontrivial, and local energy decay holds globally.

One key intermediate step of our approach here is to establish a weaker bound, which we call *two point local energy decay*, see (1.6) below. This has the advantage that it allows for nonreal eigenvalues; however, it prohibits nonzero resonances in the symmetric case.

Our motivation for this work is that the local energy decay estimates have emerged in recent years as the core decay bounds for asymptotically flat operators. As examples we note results such as [19], which assert that local energy decay implies global Strichartz estimates or the results in [29], [20] which show that sharp pointwise decay bounds can also be obtained as a consequence of local energy decay.

Another motivation is that we seek to lay out the groundwork for a similar analysis in black hole space-times, where trapping does occur. A subsequent article will be devoted to the black hole case.

1.1. The geometric setup. We consider Lorentzian manifolds (\mathbb{R}^4, g) parametrized by coordinates $(t, x) \in \mathbb{R} \times \mathbb{R}^3$. We assume that g has signature $(3, 1)$. We set $r = |x|$. On such manifolds we consider wave operators of the form

$$(1.1) \quad P(t, x, D) = \square_{A, g} + V(t, x), \quad \square_{A, g} = (D_\alpha + A_\alpha)g^{\alpha\beta}(D_\beta + A_\beta)$$

where A and V are complex valued, and $D_\alpha = (1/i)\partial_\alpha$. We assume the metric g satisfies a weak asymptotic flatness condition and is either stationary or slowly varying in time. We also put similar conditions on the lower order terms A and V . To describe the class of metrics as well as the asymptotic flatness assumption we use the class of spaces $l^p L^q$ where the l^p norm is taken with respect to nonnegative integers indexing spatial dyadic regions, and the L^q norm is taken over dyadic regions. Then we define the class of norms for functions $q(t, x)$:

$$(1.2) \quad \|q\|_k = \sum_{|\alpha| \leq k} \|\langle x \rangle^{|\alpha|} \partial^\alpha q\|_{l^1 L^\infty}.$$

We will measure the asymptotic flatness of the metric using the norm

$$\|(h, A, V)\|_{AF} = \|h\|_2 + \|\langle x \rangle A\|_1 + \|\langle x \rangle^2 V\|_0.$$

We use this to introduce the class of asymptotically flat operators P by comparing with the Minkowski metric m as follows:

Definition 1.1. *We say that the operator P is asymptotically flat (AF) if $\|(g - m, A, V)\|_{AF}$ is finite.*

To fix the parameters our estimates and results will depend on, we set M_0, R_0 so that

$$(1.3) \quad \|(g - m, A, V)\|_{AF} \leq M_0, \quad \|(g - m, A, V)\|_{AF, > R_0} \leq \mathbf{c} \ll 1,$$

where the subscript $> R_0$ above restricts the norm to the cylinder $\{r > R_0\}$. Here the implicit constant \mathbf{c} is small but universal, in that it does not depend on any of the other parameters in our problem. As a quantitative way of measuring the decay of the AF norm at infinity we will use a frequency envelope $\{c_k\}_{k \geq \log_2 R_0}$ with the following properties:

- Dyadic bound: $\|(g - m, A, V)\|_{AF(\{r \approx 2^k\})} \lesssim c_k$.
- Summability: $\sum_k c_k \lesssim \mathbf{c}$.
- Slowly varying: $c_k/c_j \leq 2^{\delta|j-k|}$ for a small universal constant δ .

The first property of AF operators P which plays a role in our analysis is stationarity:

Definition 1.2. *a) We say that P is stationary if (g, A, V) are time independent.*

b) We say that P is ϵ -slowly varying if there exists a decomposition

$$(g - m, A, V) = (g_1, A_1, V_1) + (g_2, A_2, V_2)$$

so that

$$\|(g_1, A_1, V_1)\|_{AF} + \|\partial_t(g_2, A_2, V_2)\|_{AF} \leq \epsilon.$$

The second global property of the operator P that plays a role in our results is its self-adjointness. Precisely, given a measure $dV = \mu(x)dx$ we can define the adjoint operator P^* with respect to $L^2(dV)$ by

$$\langle Pu, v \rangle_\mu = \langle u, P^*v \rangle_\mu.$$

Here one could consider two standard scenarios:

- (i) \square_g is in the Laplace-Beltrami form, and $dV = \sqrt{g}dx$.
- (ii) \square_g is in divergence form, and $dV = dx$.

But the first case reduces to the second after conjugating by $g^{\frac{1}{4}}$, without affecting anything in our analysis. Thus from here on we assume that we are in the latter case. We remark that P^* satisfies the same asymptotic flatness bounds as P .

Definition 1.3. *a) We say that the operator P is symmetric if $P^* = P$, i.e. A and V are real.*

b) We say that P is ϵ -almost symmetric if $\|(0, \Im A, \Im V)\|_{AF} \lesssim \epsilon$.

Next, we state a quantitative form of nontrapping for the space-time metric g , which will play a major role in the present work:

Definition 1.4. *We say that the asymptotically Lorentzian metric g is nontrapping if*

- (1) *The vector field ∂_t is uniformly time-like.*

(2) There exists $T_0 > 0$ such that all null geodesics spend a time no longer than T_0 inside $B(0, R_0)$.

If g were stationary, then a compactness argument shows that the above quantitative condition follows from a similar qualitative condition. However, the quantitative version of nontrapping is needed for time dependent operators P .

We also remark that the first condition above is an ellipticity condition for the operator $P_0 = P|_{D_t=0}$. The asymptotic flatness conditions already guarantee that this holds outside of a compact set. If in addition the metric is ϵ -slowly varying then we can obtain this first condition everywhere from the second:

Lemma 1.5. *Let g be an ϵ -slowly varying asymptotically flat metric which satisfies the condition (2) in Definition 1.4. If ϵ is sufficiently small,*

$$\epsilon \ll_{R_0, T_0, M_0} 1,$$

then the vector field ∂_t is uniformly timelike.

Proof. If that were not the case, then there is a point where (t_0, x_0) where $\langle \partial_t, \partial_t \rangle_g = 0$. Then we can use the ϵ -approximate conservation of $\langle \partial_t, \partial_t \rangle_g$ along the null geodesic with initial direction ∂_t at (t_0, x_0) to derive a contradiction, because one must have $\langle \partial_t, \partial_t \rangle_g \approx -1$ at points along the null geodesic by the time T_0 , when it reaches the exterior region $|x| > R_0$. \square

1.2. Local energy norms. The space-time local energy norms are defined as follows:

$$(1.4) \quad \|u\|_{LE} = \|u\|_{\ell^\infty L^{2, -\frac{1}{2}}}, \quad \|u\|_{LE^1} = \|(\partial u, \langle x \rangle^{-1} u)\|_{LE}.$$

Here $\partial = (\partial_\alpha)_{\alpha=0, \dots, 3}$ and $\|w\|_{L^{q, \sigma}} = \|w\|_{L^q(\langle x \rangle^\sigma dx)}$. We shall reserve the notation ∇ for $\nabla = (\partial_j)_{j=1, 2, 3}$. For the source term we use the dual space

$$\|F\|_{LE^*} = \|F\|_{\ell^1 L^{2, \frac{1}{2}}}.$$

For the corresponding analysis of the resolvent we also need the related norms at fixed time; these are denoted by \mathcal{LE} , \mathcal{LE}^1 , \mathcal{LE}^* . By LE_0 , \mathcal{LE}_0 , etc. we denote the closure of C_0^∞ in these spaces; this corresponds to replacing ℓ^∞ by c^0 in the definition above. For later purposes when time derivatives will be replaced by a spectral parameter ω , we also define

$$\mathcal{LE}_\omega^1 = \mathcal{LE}^1 \cap |\omega|^{-1} \mathcal{LE}, \quad \dot{H}_\omega^1 = \dot{H}^1 \cap |\omega|^{-1} L^2.$$

Our main goal is to understand which space-times (\mathbb{R}^4, g) and operators P satisfy *local energy decay* estimates:

Definition 1.6. *We say that local energy decay holds for a nontrapping asymptotically flat operator P if the following estimate holds:*

$$(1.5) \quad \|u\|_{LE^1[0, T]} + \|\partial u\|_{L^\infty L^2[0, T]} \lesssim \|\partial u(0)\|_{L^2} + \|Pu\|_{LE^* + L^1 L^2[0, T]}.$$

Estimates of this type have a long history, and were first introduced in the work of Morawetz [21, 22, 23] for the Minkowski space-time; for further references we refer the reader to [1], [4], [9], [10], [11], [17, 18], [24], [25], [27], [28]. As starting point for this work we use the result in [19], which conveniently uses the same definition of asymptotic flatness as here.

Theorem 1.7 ([19]). *Assume that P is a small AF perturbation of \square . Then local energy decay holds for P .*

Implicit in the above estimate is a global-in-time uniform energy bound. A qualitatively weaker version of the local energy decay is what we call the *two point local energy decay* estimate. This allows for spatially localized exponentially growing or decaying solutions:

Definition 1.8. *We say that two point local energy decay holds for a nontrapping asymptotically flat operator P if the following estimate holds:*

$$(1.6) \quad \|u\|_{LE^1[0, T]} + \|\partial u\|_{L^\infty L^2[0, T]} \lesssim \|\partial u(0)\|_{L^2} + \|\partial u(T)\|_{L^2} + \|Pu\|_{LE^* + L^1 L^2[0, T]}.$$

In general this estimate also does not exclude nonzero resonances, discussed below, except in the special case of symmetric operators P . However, it does preclude stationary or nearly stationary spatially localized solutions. An even weaker estimate whose role is to only preclude stationary solutions is a condition introduced in [20], called *stationary local energy decay*:

Definition 1.9. We say that stationary local energy decay holds for a nontrapping asymptotically flat operator P if the following estimate holds:

$$(1.7) \quad \|u\|_{LE^1[0,T]} + \|\partial u\|_{L^\infty L^2[0,T]} \lesssim \|\partial u(0)\|_{L^2} + \|\partial u(T)\|_{L^2} + \|\partial_0 u\|_{LE[0,T]} + \|Pu\|_{LE^* + L^1 L^2[0,T]}.$$

1.3. Energy estimates. As defined in the previous section, uniform energy bounds of the form

$$(1.8) \quad \|\partial u\|_{L^\infty L^2} \lesssim \|\partial u(0)\|_{L^2} + \|Pu\|_{L^1 L^2}$$

are included as a subset of the local energy decay estimates. Further, they provide a direct connection between the more robust two point local energy decay bound (1.6) and the local energy bound (1.5).

For this and other reasons, it is useful to briefly discuss the energy conservation properties for the wave equation $Pu = f$. This is most easily done in the symmetric stationary case, where we have a conserved energy functional for the homogeneous problem, namely

$$(1.9) \quad E(u) = \int_{\mathbb{R}^3} \bar{u} P_0 u - g^{00} |\partial_t u|^2 dx, \quad P_0 = P|_{D_t=0}.$$

For the inhomogeneous problem $Pu = f$, we have the relation

$$(1.10) \quad \frac{d}{dt} E(u) = 2\Re \int_{\mathbb{R}^3} \bar{f} \partial_t u dx.$$

The difficulty in using this energy is that it is not necessarily positive definite. However, in view of condition (1) in Definition 1.4, the operator P_0 is elliptic and asymptotically flat, so the obstruction to that is localized in a compact set, i.e.

$$(1.11) \quad \|(u, \partial_t u)\|_{H^1 \times L^2}^2 \lesssim E(u) + \|u\|_{L_{comp}^2}^2.$$

For a further discussion of this case we refer the reader to the last section of the paper.

If we drop the stationarity assumption, then the energy functional defined above becomes time dependent and is no longer conserved. If we also drop the symmetry assumption then it is natural to work with the (time dependent) energy associated to the symmetric part of P . This leads us to a straightforward almost conservation property

Lemma 1.10. Assume that P is asymptotically flat, ϵ -slowly varying and ϵ -almost symmetric. Then we have the energy conservation relation

$$(1.12) \quad E(u(T)) = E(u(0)) + O(\epsilon) \|u\|_{LE^1[0,T]}^2$$

for solutions to the homogeneous equation $Pu = 0$.

The proof of this is straightforward and is left for the reader.

2. MAIN RESULTS

2.1. The stationary case. Here we discuss the easier case of operators P which are stationary, where we can provide a full spectral description for our local energy decay problem. Later, we will use this case as a guide for what to expect in the more general case.

The results here are more in line with earlier work on the subject, at least in the self-adjoint case, see for instance [7], [8], [29]. However, for clarity and further reference it is perhaps of interest to provide a full, self-contained set of results.

To set the notations and introduce the corresponding spectral problems, we consider special solutions to the homogeneous wave equation $P(x, D)u = 0$ that have the form $u = e^{i\omega t} u_\omega(x)$. Then we are led to the quadratic eigenvalue problem:

$$(2.1) \quad P_\omega u_\omega = 0,$$

where P_ω is the operator

$$(2.2) \quad P_\omega = \Delta_{g,A} + W(x, D_x) + \omega B(x, D_x) + \omega^2 g^{00}$$

with

$$\begin{aligned}\Delta_{g,A} &= (D+A)_j g^{jk} (D+A)_k, \\ W(x, D_x) &= A_0 g^{0k} (D_k + A_k) + (D_k + A_k) A_0 g^{0k} + g^{00} A_0^2 + V(x), \\ B(x, D_x) &= g^{0k} (D_k + A_k) + (D_k + A_k) g^{0k} + 2A_0 g^{00}.\end{aligned}$$

We remark that if P is self-adjoint and ω is real, then P_ω is self-adjoint. However, the above eigenvalue problem is not self-adjoint so there is no reason for ω to be real, and in fact if $\omega \notin \mathbb{R}$ then one can expect there to be nontrivial root space structure associated to u_ω .

The resolvent R_ω is defined as the inverse of P_ω whenever the inverse exists. It can also be defined as the Fourier-Laplace transform of the wave group as follows. Suppose that u solves the homogeneous wave equation

$$Pu = 0, \quad u(0) = 0, \quad g^{00}u_t(0) = f.$$

Then the resolvent R_ω is given by

$$(2.3) \quad R_\omega f = \int_0^\infty e^{-it\omega} u(t) dt.$$

A priori the local theory for the wave equation yields an exponential bound of the form

$$(2.4) \quad \|\partial u(t)\|_{L^2} \lesssim e^{\beta t} \|f\|_{L^2}.$$

This implies that the resolvent R_ω is defined at least in the halfspace $\{\Im \omega < -\beta\}$, with an uniform bound

$$(2.5) \quad \|R_\omega\|_{L^2 \rightarrow \dot{H}_\omega^1} \lesssim |\Im \omega + \beta|^{-1}, \quad \Im \omega < -\beta.$$

Further, one can invert the Fourier transform and express the wave evolution in terms of the resolvent as

$$(2.6) \quad u(t) = \int_{\Im \omega = -\sigma} e^{i\omega t} R_\omega f d\omega, \quad \sigma > \beta.$$

The local energy bounds for the forward evolution associated to the operator P are closely related to uniform energy bounds for the forward evolution. As in the above discussion, this corresponds to an extension of the resolvent into the lower half space. In relation to this, we have the following standard result:

Proposition 2.1. *a) The resolvent $R_\omega : L^2 \rightarrow \dot{H}_\omega^1$ admits a meromorphic extension to the lower half-plane $H = \{\Im \omega < 0\}$.*

b) If uniform energy bounds hold for the forward evolution associated to P then the resolvent is holomorphic in H and satisfies

$$(2.7) \quad \|R_\omega\|_{L^2 \rightarrow \dot{H}_\omega^1} \lesssim |\Im \omega|^{-1}, \quad \omega \in H.$$

c) Conversely, if the above bound holds then the solutions u to the homogeneous equation $Pu = 0$ satisfy the energy bound

$$(2.8) \quad \|\partial u(t)\|_{L^2} \lesssim (1+t) \|\partial u(0)\|_{L^2}.$$

The proof of this theorem is fairly standard and the details are left for the reader. For part (a) it suffices to observe that the operators P_ω are uniformly elliptic for ω in a compact subset of H . Part (b) follows by direct integration in (2.3) and represents one direction (the easy one) in the Hille-Yosida theorem.

Part (c) is a consequence of the averaged L^2 bound

$$\|e^{-\sigma t} \partial u\|_{L^2_{x,t}} \lesssim \sigma^{-1} \|e^{-\sigma t} Pu\|_{L^2_{x,t}}, \quad \sigma > 0,$$

which follows by Plancherel from (2.7). For the forward homogeneous initial value problem $Pu = 0$ this gives by truncation

$$\|e^{-\sigma t} \partial u\|_{L^2_{x,t}([0,\infty) \times \mathbb{R}^3)} \lesssim \sigma^{-1} \|\partial u(0)\|_{L^2} \quad \sigma > 0.$$

One then chooses $\sigma = t^{-1}$, and the energy bound at time t follows from the averaged bound on the time interval $[t-1, t]$. Most likely one should be able to improve the polynomial growth rate to $(1+t)^{\frac{1}{2}}$; it is not clear to us what the correct threshold is.

The poles of the resolvent R_ω in the lower half-space H are zero eigenfunctions for the corresponding elliptic operators P_ω . By Fredholm theory the associated generalized eigenspaces are finite dimensional.

Further, also due to the ellipticity of P_ω , the associated eigenfunctions and generalized eigenfunctions are spatially localized and decay exponentially at infinity. These correspond to exponentially growing solutions to the homogeneous wave equation $Pu = 0$. Thus we have identified a first obstruction to both uniform energy bounds and local energy decay:

Definition 2.2. *A negative eigenfunction for P is a function $u_\omega \in L^2$ so that $P_\omega u_\omega = 0$ with $\omega \in H$.*

We also observe that one can similarly define positive eigenfunctions and associate them to exponentially decreasing solutions to the wave equation. This can also be viewed as the identical construction but with reversed time.

We now consider the interpretation of the local energy bounds in terms of corresponding resolvent bounds. For this we have the following result:

Proposition 2.3. *Local energy decay holds for the operator P if and only if the resolvent R_ω satisfies a uniform bound*

$$(2.9) \quad \|R_\omega\|_{\mathcal{LE}^* \rightarrow \mathcal{LE}^1_\omega} \lesssim 1, \quad \Im \omega < 0.$$

This is a fairly straightforward result, which is more or less a consequence of the Plancherel formula. The proof is in Section 8. See also [29].

This result leads us to a second obstruction to local energy decay, which is obtained by taking the limit of the resolvent as ω approaches the real axis. There are two cases to consider, depending whether the limit is zero or not.

In the nonzero case, in the limit we obtain a boundary condition at infinity, called the outgoing radiation condition or the Sommerfeld condition. This takes the form

$$(2.10) \quad 2^{-j/2} \|(\partial_r + i\omega)u\|_{L^2(|x| \approx 2^j)} \rightarrow 0.$$

Thus we define

Definition 2.4. $\omega \in \mathbb{R} \setminus \{0\}$ *is an embedded resonance for P if there exists a function $u_\omega \in \mathcal{LE}^1_\omega$, satisfying the outgoing radiation condition (2.10), so that $P_\omega u_\omega = 0$.*

We carefully exclude the case $\omega = 0$ here. It is not too difficult to show that in polar coordinates (r, θ) the embedded resonances must have the asymptotics

$$u_\omega(r, \theta) = v(\theta)r^{-1}e^{i\omega r} + o(r^{-1}), \quad v \neq 0$$

where the condition $v \neq 0$ is a consequence of unique continuation results, see [13]. We observe that solutions of the form $e^{i\omega t}u_\omega$ cannot be directly used as obstructions to local energy decay or energy estimates, as they do not have finite energy. However, at least for nice enough metrics, one can consider truncated versions of them, of the form

$$\tilde{u}_\omega = \chi(t - r)e^{i\omega t}u_\omega$$

with a smooth cutoff function χ which selects the positive real line and using suitable Regge-Wheeler type coordinates. Then a direct computation yields the asymptotic relations

$$\|\partial \tilde{u}_\omega(t)\|_{L^2} \approx t^{\frac{1}{2}}, \quad \|P\tilde{u}_\omega(t)\|_{L^2} \in L^1.$$

This shows that embedded resonances are obstructions to both uniform energy bounds and to local energy decay. The generated energy growth is $t^{\frac{1}{2}}$. This is localized in the outgoing region but not concentrated near a light cone.

It may also be interesting to observe that, as part of the proof of our main result in Theorem 2.12, we establish the following:

Proposition 2.5 (Limiting Absorption Principle). *The following statements are equivalent for stationary, asymptotically flat operators P and real nonzero ω_0 :*

- a) ω_0 is not a resonance.
- b) The estimate

$$(2.11) \quad \|u\|_{\mathcal{LE}^1_{\omega_0}} \lesssim \|P_{\omega_0}u\|_{\mathcal{LE}^*}$$

holds for all $u \in \mathcal{LE}^1_{\omega_0}$ satisfying the outgoing resonance condition (2.10).

c) The resolvent bound (2.9) holds uniformly for $\omega \in H$ near ω_0 , and the limit

$$R_{\omega_0}f = \lim_{H \ni \omega \rightarrow \omega_0} R_{\omega}f$$

exists uniformly on compact sets and satisfies the outgoing resonance condition (2.10).

Furthermore, these properties are stable with respect to small, stationary, asymptotically flat perturbations of P .

A significant simplification occurs in the case when P is symmetric:

Theorem 2.6 ([13]). *Assume that P is a symmetric, asymptotically flat wave operator. Then there are no nonzero resonances for P .*

This is a classical result but, in the context we are using, it was proved in [13]. We emphasize here the self-adjointness requirement, without which the result is no longer true. However, assuming nontrapping, it is stable with respect to small non-self-adjoint perturbations.

Remark 2.7. The proof of this result has two complementary steps:

- (1) Use the symmetry and the outgoing radiation condition (2.10) to improve the decay of a resonance $u_{\omega} \in \mathcal{LE}^1$ to $u_{\omega} \in \mathcal{LE}_0^1$.
- (2) Use Carleman estimates to establish a unique continuation type result from infinity which shows that $u_{\omega} \in \mathcal{LE}_0^1$ implies $u_{\omega} = 0$.

The second step above no longer uses the symmetry, and we will exploit this fact in the nonstationary part of the paper.

The second limit we need to consider is as the spectral parameter ω goes to zero. In that case the outgoing radiation condition is no longer meaningful, so we set

Definition 2.8. *A function u_0 is a zero resonance/eigenfunction for P if $u_0 \in \mathcal{LE}_0$ and $P_0 u_0 = 0$.*

The condition $u_0 \in \mathcal{LE}_0$ excludes asymptotically constant functions. Then by standard elliptic analysis one can show that zero resonances have better decay at infinity and simpler asymptotics,

$$u(r, \omega) = cr^{-1} + o(r^{-1}), \quad \nabla u = O(r^{-2}).$$

We remark that zero resonances for P_0 correspond to stationary finite energy solutions for P . These have infinite local energy norm. Generically the constant c in the above asymptotics is nonzero. However, unlike the case of resonances, here we may encounter a stronger decay (though no faster than polynomial, due to unique continuation results from infinity, see [12]). Also, the homogeneous equation $Pu = 0$ may have further solutions with polynomial growth in time.

In a direct fashion, one would relate the fact that zero is not an eigenvalue/resonance to a bound of the form

$$(2.12) \quad \|u\|_{\mathcal{LE}^1} \lesssim \|P_0 u\|_{\mathcal{LE}^*}, \quad u \in \mathcal{LE}_0^1.$$

However, since P_0 is a classical elliptic operator, its behavior at infinity is perturbative off the Laplacian. But the Laplacian satisfies a larger family of weighted estimates, so the choice of weights at infinity here is not so important. This motivates the following definition:

Definition 2.9. *We say that P satisfies a zero resolvent bound (or zero nonresonance condition) if*

$$(2.13) \quad \|u\|_{\dot{H}^1} \leq K_0 \|P_0 u\|_{\dot{H}^{-1}}.$$

By analogy with Proposition 2.5, we have the following result, which is proved in Section 6:

Proposition 2.10 (Zero resolvent bound). *The following statements are equivalent for stationary, asymptotically flat operators P :*

- a) 0 is not a resonance or eigenvalue.
- b) The estimate (2.12) holds for all $u \in \mathcal{LE}_0^1$.
- c) The estimate (2.13) holds for all $u \in \dot{H}^1$.

d) The resolvent bound (2.9) holds uniformly for $\omega \in H$ near 0, and the limit

$$R_0 f = \lim_{H \ni \omega \rightarrow 0} R_\omega f \in \mathcal{LE}_0^1$$

exists uniformly on compact sets.

e) The stationary local energy decay bound (1.7) holds.¹

Furthermore, these properties are stable with respect to small, stationary, asymptotically flat perturbations of P .

The discussion above of eigenvalues and resonances is local in nature, so in order to obtain global information out of it, it would be very convenient if we were able to restrict these considerations to a compact set. This is where the nontrapping assumption first comes in. Precisely, we have the following high frequency local energy decay bound:

Theorem 2.11 (High frequency local energy decay). *For any nontrapping asymptotically flat operator P we have the estimate:*

$$(2.14) \quad \|u\|_{LE^1[0,T]} + \|\partial u\|_{L^\infty L^2[0,T]} \lesssim \|\partial u(0)\|_{L^2} + \|\langle x \rangle^{-2} u\|_{LE} + \|Pu\|_{LE^* + L^1 L^2[0,T]},$$

where the implicit constant is uniform in T .

We remark that stationarity plays no role in this theorem. However, as we shall see in the proof, in the stationary case this yields good resolvent bounds outside a compact set of frequencies. Now we can state our main result for stationary operators:

Theorem 2.12. a) *Let P be stationary, nontrapping and asymptotically flat. Then local energy decay holds for P if and only if P has no negative eigenvalues or real resonances.*

b) *Let P be stationary, symmetric, nontrapping and asymptotically flat. Then local energy decay holds for P if and only if P has no negative eigenvalues or zero eigenvalues/resonances.*

We remark that this result is also stable with respect to small asymptotically flat perturbations. In particular, in the symmetric case one can continue the property of absence of negative eigenvalues along continuous families of operators $P(h)$ for as long as no zero eigenvalues/resonances occur (see Proposition 2.14 below).

In what follows we take a closer look at the symmetric case if we retain the no zero resonance assumption (2.13), but drop the requirement that there are no negative eigenfunctions. As the eigenfunctions with negative imaginary part are isolated and localized to a compact² set, it follows that there are finitely many of them. Further, by Fredholm theory, each of them has an associated finite dimensional invariant subspace. The sum of these subspaces is associated to solutions which grow exponentially in time. One can argue in a similar manner for eigenvalues with positive imaginary part (which are the complex conjugates of the ones with negative imaginary part). This heuristic reasoning leads to the following theorem:

Theorem 2.13. *Let P be symmetric, stationary, nontrapping, asymptotically flat, and satisfying the zero resolvent bound (2.13). Then there is a direct sum decomposition of the energy space into invariant subspaces*

$$(2.15) \quad \mathcal{E} = S_0 + S^+ + S^-,$$

with S^+ and S^- of equal finite dimension, so that the wave flow on the subspaces S^+ , S^0 and S^- is as follows:

- a) *Solutions in S^+ are spatially localized and grow exponentially in time.*
- b) *Solutions in S^0 are uniformly bounded and satisfy local energy decay.*
- c) *Solutions in S^- are spatially localized and decay exponentially in time.*

Likely this result extends fully to the non-self-adjoint case under a no resonance assumption; we leave this as an open problem. Further, this result has a nice stability property with respect to P . Precisely, as a consequence of Propositions 2.5, 2.10 and 9.3 we have the following:

¹In the proof we work with (6.1) rather than (1.7). We note, however, that the latter follows from the former for bounded ranges of frequencies and that, by (2.14), the bound is generically true for high frequencies.

²This is a consequence of the nontrapping assumption, and part of the proof of the last theorem.

Proposition 2.14. *Let $P(h)$ be a family of stationary, symmetric, nontrapping and asymptotically flat wave operators which is continuous in the AF topology. Suppose that for each h there are no zero eigenvalues/resonances for $P(h)$. Then the (equal) dimensions of the invariant subspaces S^\pm for $P(h)$ are independent of h .*

2.2. The nonstationary case. Our main goal here is to show that the results in the stationary case extend without any change to almost symmetric, almost stationary operators. We remark that this includes small AF perturbations of stationary metrics, but it is certainly not limited to that. While the results here are very similar³ to those in the stationary case, our approach requires some new ideas, in part to account for the need to have good bounds of the implicit constants in the estimates in terms of our control parameters R_0 , M_0 , T_0 and K_0 .

Our starting point here is a local energy decay estimate with bounds in terms of initial and final energies:

Theorem 2.15 (Two point local energy decay). *Let P be an ϵ -slowly varying, nontrapping and asymptotically flat operator, which satisfies the zero nonresonance condition (2.13) for some $K_0 > 0$. Suppose that ϵ is small enough*

$$\epsilon \ll_{R_0, M_0, T_0, K_0} 1.$$

Then we have the estimate:

$$(2.16) \quad \|u\|_{LE^1[0, T]} + \|\partial u\|_{L^\infty L^2[0, T]} \lesssim \|\partial u(0)\|_{L^2} + \|\partial u(T)\|_{L^2} + \|Pu\|_{LE^* + L^1 L^2[0, T]},$$

where the implicit constant is uniform in T and depends only on R_0 , M_0 , T_0 and K_0 .

We remark here one key advantage of working with the two point local energy decay, namely that it does not see the negative eigenvalues of P or nonzero resonances. Because of this, such an estimate is not limited to the (almost) symmetric case. However, in order to use this estimate to get bounds on the energy growth rate, we do need to use an almost conservation of energy, which requires the almost symmetry condition. Under an additional almost symmetry assumption, from this theorem one gains the following simple dichotomy:

Theorem 2.16 (Exponential dichotomy). *Let P be an ϵ -slowly varying, ϵ -almost symmetric, nontrapping and asymptotically flat operator, which satisfies the zero nonresonance condition (2.13) for some $K_0 > 0$. Suppose that ϵ is small enough*

$$\epsilon \ll_{R_0, M_0, T_0, K_0} 1.$$

Then there exists an $\alpha_0 > 0$, depending on the parameters R_0, M_0, T_0, K_0 , such that any solution u to the inhomogeneous equation $Pu = f$ with $u[0] \in \mathcal{E}$ and $f \in LE^ + L^1 L^2$ has one of the following two properties:*

(i) *Exponential growth for large t :*

$$(2.17) \quad \|\partial u(t)\|_{L^2} \geq ce^{\alpha_0 t} (\|\partial u(0)\|_{L^2} + \|Pu\|_{LE^* + L^1 L^2[0, \infty)}) \quad t > t_0,$$

(ii) *Global bound and local energy decay estimate:*

$$(2.18) \quad \|u\|_{LE^1[0, T]} + \|\partial u\|_{L^\infty L^2} \lesssim \|\partial u(0)\|_{L^2} + \|Pu\|_{LE^* + L^1 L^2[0, \infty)}.$$

Heuristically, the role played by the (almost) symmetry is to eliminate the nonzero resonances, at least in the stationary case. Indeed, the proof of this result is a time dependent take on the classical proof of the absence of embedded resonances for the stationary problem.

Finally, for a more accurate result we can also take into account the exponentially decaying solutions.

Theorem 2.17 (Exponential trichotomy). *Let P satisfy all of the assumptions of Theorem 2.16. Then there exist a direct sum decomposition*

$$\dot{H}^1 \times L^2 = S^+ + S^0 + S^-$$

with κ dimensional linear subspaces S^\pm , and corresponding commuting projections P^0 , P^\pm , such that if $\mathcal{F}(t)$ denotes the flow of the homogeneous equation $Pu = 0$ on $\dot{H}^1 \times L^2$ with data at $t = 0$ then:

$$(2.19) \quad \|\mathcal{F}(t)P^- \mathcal{F}^{-1}(s)\|_{\dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2} \leq Ce^{-\alpha(t-s)}, \quad t \geq s \geq 0,$$

$$(2.20) \quad \|\mathcal{F}(s)P^+ \mathcal{F}^{-1}(t)\|_{\dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2} \leq Ce^{-\alpha(t-s)}, \quad t \geq s \geq 0,$$

$$(2.21) \quad \|\mathcal{F}(t)P^0 \mathcal{F}^{-1}(s)\|_{\dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2} \leq C, \quad \text{all } t, s \geq 0.$$

³Except for the almost symmetry hypothesis, which we need in order to avoid resonances related issues.

Here $\mathcal{F}^{-1}(s)$ is the algebraic inverse map of $\mathcal{F}(s)$ on $\dot{H}^1 \times L^2$ (its existence as a bounded operator is easily established via standard energy estimates).

By the previous theorem, the solutions with data in S^0 will also satisfy local energy decay bounds. To unravel a bit what this last Theorem says let $P^\pm(t) = \mathcal{F}(t)P^\pm\mathcal{F}^{-1}(t)$ be the flow projections onto the image $S^\pm(t) = \mathcal{F}(t)S^\pm$, set $P^0(t) = 1 - P^+(t) - P^-(t)$ with corresponding image $S^0(t)$, and let $\mathcal{F}(t, s) = \mathcal{F}(t)\mathcal{F}^{-1}(s)$ denote the propagator from times $s \rightarrow t$. Then estimates (2.19)–(2.21) can be written in the equivalent form (for a different C):

$$(2.22) \quad \|\mathcal{F}(t, s)\|_{S^-(s) \rightarrow S^-(t)} \leq Ce^{-\alpha(t-s)}, \quad t \geq s \geq 0,$$

$$(2.23) \quad \|\mathcal{F}(t, s)\|_{S^+(s) \rightarrow S^+(t)} \geq C^{-1}e^{\alpha(t-s)}, \quad t \geq s \geq 0,$$

$$(2.24) \quad C^{-1} \leq \|\mathcal{F}(t, s)\|_{S^0(s) \rightarrow S^0(t)} \leq C, \quad \text{all } t, s \geq 0.$$

$$(2.25) \quad \|P^{\pm,0}(t)\|_{\dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2} \leq C, \quad \text{all } t, s \geq 0.$$

In other words the flow along $S^-(t)$ is uniformly contracting at an exponential rate forward in time, while the flow along $S^+(t)$ is uniformly expanding at an exponential rate forward in time. On the other hand $S^0(t)$ is a center manifold on which the flow has uniformly bounded energy both forward and backward in time. According to Theorem 2.15 the flow along $S^0(t)$ is therefore uniformly dispersive with bounds on $\|\phi\|_{LE[s,t]}$ in terms of $\|\partial\phi(s)\|_{L^2} \approx \|\partial\phi(t)\|_{L^2}$ for any set of times $s, t \geq 0$. Finally, there is a bit of additional information given by (2.25) which says that the time dependent subspaces $S^{\pm,0}(t)$ are “uniformly transverse”.

2.3. An overview of the paper. The bulk of the article is devoted to the proof of the various local energy decay bounds. This is developed as follows:

In Section 3 we expand upon the result of [19], see Theorem 1.7, which asserts that local energy decay holds for small asymptotically flat perturbations of the d’Alembertian. Our goal is to work with operators P which have this property outside a cylinder and to describe two efficient ways of truncating the estimates to such an exterior region.

Section 4 is devoted to the proof of the high frequency local energy bound in Theorem 2.11. This is a classical positive commutator estimate, with the added difficulty that we do not use any energy conservation property.

The following two sections contain the remaining two blocks in the proof of the two point local energy estimate in Theorem 2.15. We begin with the medium frequencies in Section 5; the analysis there is based on Carleman estimates and can be viewed as a time dependent version of the Carleman estimates for the resolvent in [13]. The low frequency analysis follows in Section 6.

The three building blocks above, namely the high, medium and low frequency analysis are assembled together in Section 7 in order to prove the two point local energy estimate in Theorem 2.15. We also give there the closely related proof of the exponential dichotomy in Theorem 2.16.

In Section 8 we consider the relation between the local energy estimates and resolvent bounds, leading to the proof of Propositions 2.3, 2.5 and 2.10. We conclude with the proof of Theorem 2.12.

Last but not least, in Section 9 we fully exploit the two point local energy decay bound. We begin with the stationary case, where we establish the exponential trichotomy in Theorem 2.13. Then we switch to the nonstationary case, and prove Theorem 2.17.

3. EXTERIOR ESTIMATES FOR ASYMPTOTICALLY FLAT PERTURBATIONS OF MINKOWSKI

There are two exterior local energy decay estimates which we will use. Our starting point is the standard local energy decay estimate for small AF perturbations of \square in Theorem 1.7, which was proved in [19]. Applying a cutoff function which selects an exterior region $\{r \geq R\}$ we directly obtain a truncated estimate

$$(3.1) \quad \|\phi\|_{LE_{>R}^1} + \|\partial\phi\|_{L^\infty_{>R}} \lesssim \|\partial\phi(0)\|_{L^2_{>R}} + \|\phi\|_{LE_R^1} + \|P\phi\|_{LE_{>R}^*}.$$

Here we explore two nontrivial ways of truncating the bound in Theorem 1.7 to an exterior region. The first will handle very low frequencies, while the second is useful in situations where one can absorb a low frequency error.

Our first truncated estimate uses only the localized L^2 norm of ∂u as the truncation error:

Proposition 3.1. *Suppose P is asymptotically flat. Then for $R > R_0$ we have*

$$(3.2) \quad \|\phi\|_{LE^1_{>R}} \lesssim \|\partial\phi(0)\|_{L^2_{>R}} + \|\partial\phi\|_{LE_R} + \|P\phi\|_{LE^*_{>R}}.$$

Proof. The restriction $R > R_0$ simply says that we are in a regime where P is a small AF perturbation of \square . Without any restriction in generality we assume that this is the case globally, with the same AF constant \mathbf{c} as in (1.3).

To prove the above bound we will apply the global LE bound for small AF perturbations of \square to a suitably chosen extension of ϕ inside the R cylinder. To define this extension we introduce the local average of ϕ adapted to the R annulus as

$$\bar{\phi}_R(t_0) = R^{-4} \int \phi(t, x) \chi\left(\frac{t-t_0}{R}, \frac{|x|}{R}\right) dx dt$$

where $\chi(t, r)$ is a smooth, nonnegative bump function supported in $[-1, 1] \times [1, 2]$, with unit integral. By integration by parts, the Schwarz inequality, and Young's inequality, we have

$$(3.3) \quad \|\partial_t^j \bar{\phi}_R\|_{L^2} \lesssim R^{-j} \|\partial\phi\|_{LE_R}, \quad j \geq 1.$$

We also have a Poincaré type inequality

$$(3.4) \quad \|\langle r \rangle^{-1} (\bar{\phi}_R - \phi)\|_{LE_R} \lesssim \|\partial\phi\|_{LE_R}.$$

Now we extend the function ϕ inside the R cylinder by

$$\phi_{ext}(t, x) = \beta_{>R}(|x|) \phi(t, x) + (1 - \beta_{>R})(|x|) \bar{\phi}_R(t),$$

where, e.g., we may take $\beta_{>R}(r) = 1 - \beta(r/R)$ with $\beta \in C^\infty(\mathbb{R}_+)$ with $\beta \equiv 1$ on $[0, 1]$ and supported in $[0, 2]$. By Theorem 1.7, it follows that

$$\|\phi_{ext}\|_{LE^1} \lesssim \|\partial\phi_{ext}(0)\|_{L^2} + \|P\phi_{ext}\|_{LE^*},$$

and its local energy norm is given by

$$\|\phi_{ext}\|_{LE^1} \approx \|\beta_{>R}\phi\|_{LE^1} + \|\bar{\phi}_R\|_{L^2} + R \|\partial_t \bar{\phi}_R\|_{L^2}.$$

On the other hand, we compute

$$P\phi_{ext} = \beta_{>R}P\phi + (1 - \beta_{>R})P\bar{\phi}_R + [P, \beta_{>R}](\phi - \bar{\phi}_R).$$

The commutator is first order and supported inside the R annulus with

$$[P, \beta_{>R}](\phi - \bar{\phi}_R) = O(R^{-1})|\partial\phi| + O(R^{-1})|\partial_t \bar{\phi}_R| + O(R^{-2})|\phi - \bar{\phi}_R|, \quad |x| \approx R.$$

Thus, using (3.3) and (3.4) respectively on each of the last two terms, we see that

$$\|[P, \beta_{>R}](\phi - \bar{\phi}_R)\|_{LE^*} \lesssim \|P\phi\|_{LE^*_{>R}} + \|\partial\phi\|_{LE_R}.$$

Moreover, we note that, as we are taking without loss of generality $\|(g - m, A, V)\|_{AF} \leq \mathbf{c}$,

$$\|(1 - \beta_{>R})P\bar{\phi}_R\|_{LE^*} \lesssim R^2 \|\partial_t^2 \bar{\phi}_R\|_{L^2} + R \|\partial_t \bar{\phi}_R\|_{L^2} + \mathbf{c} \|\bar{\phi}_R\|_{L^2}.$$

Using (3.3), using $\mathbf{c} \ll 1$ in order to bootstrap, and combining we have that

$$\|\beta_{>R}\phi\|_{LE^1} + \|\bar{\phi}_R\|_{L^2} \lesssim \|\partial\phi_{ext}(0)\|_{L^2} + \|P\phi\|_{LE^*_{>R}} + \|\partial\phi\|_{LE_R}.$$

Observing that the Schwarz inequality yields

$$\|\partial\phi_{ext}(0)\|_{L^2} \lesssim \|\partial\phi(0)\|_{L^2_{>R}} + \|\partial\phi\|_{LE_R}$$

then completes the proof. \square

Our second truncated estimate uses instead a localized L^2 norm of u as the truncation error. The price to pay is that we need the energy of u at both the initial and the final time. This bound will be used later in the proof of the Carleman estimate (5.4) in Proposition 5.2.

Proposition 3.2. *Suppose P is asymptotically flat. Then for $R \geq R_0$ we have:*

$$(3.5) \quad \|u\|_{LE^1_{>R}} \lesssim \|\partial u(T)\|_{L^2_{>R}} + \|\partial u(0)\|_{L^2_{>R}} + \|Pu\|_{LE^*_{>R}} + R^{-1} \|u\|_{LE_R}.$$

Proof. This is a multiplier estimate, which is similar to an estimate for the corresponding Schrödinger equation in [16], and is related to the earlier work [3].

Setting $\square_g = \square_{0,g}$, we begin by noticing that if $Q(x, D_x)$ is self-adjoint, then

$$(3.6) \quad \frac{d}{dt} \left\{ 2\Re \langle g^{00} D_t u, Qu \rangle + 2\Re \langle g^{j0} D_j u, Qu \rangle \right\} \\ = -2\Im \langle \square_g u, Qu \rangle + \langle i[D_j g^{jk} D_k, Q]u, u \rangle + 2\Im \langle [Q, g^{j0} D_j]u, D_t u \rangle - \langle i[Q, g^{00}]D_t u, D_t u \rangle.$$

Here, the inner product is that of $L^2(\mathbb{R}^3)$. We set

$$Q = \beta(r) f(r) \frac{x_l}{r} g^{lm} D_m + D_m \beta(r) f(r) \frac{x_l}{r} g^{lm}.$$

Here $f(r) = r/(r + 2^j)$ where $2^j \geq R$, and β is a smooth, increasing function so that $\beta \equiv 0$ for $r \leq R$ and $\beta \equiv 1$ for $r \geq 2R$. Then

$$[D_j g^{jk} D_k, \beta(r) f(r) \frac{x_l}{r} g^{lm} D_m] = [D_j g^{jk} D_k, \beta(r)] f(r) \frac{x_l}{r} g^{lm} D_m + \beta(r) [D_j g^{jk} D_k, f(r) \frac{x_l}{r} g^{lm} D_m] \\ [D_j g^{jk} D_k, D_m \beta(r) f(r) \frac{x_l}{r} g^{lm}] = [D_j g^{jk} D_k, D_m f(r) \frac{x_l}{r} g^{lm}] \beta + D_m f(r) \frac{x_l}{r} g^{lm} [D_j g^{jk} D_k, \beta(r)].$$

Using that

$$[D_j g^{jk} D_k, \beta] = \frac{2}{i} D_j g^{jk} \frac{x_k}{r} \beta'(r) + \partial_j \left(g^{jk} \beta'(r) \frac{x_k}{r} \right) \\ = \frac{2}{i} g^{jk} \beta'(r) \frac{x_j}{r} D_k - \partial_j \left(g^{jk} \beta'(r) \frac{x_k}{r} \right),$$

we have

$$[D_j g^{jk} D_k, \beta(r)] f(r) \frac{x_l}{r} g^{lm} D_m + D_m f(r) \frac{x_l}{r} g^{lm} [D_j g^{jk} D_k, \beta(r)] \\ = \frac{4}{i} D_j g^{jk} \frac{x_k}{r} \beta'(r) f(r) \frac{x_l}{r} g^{lm} D_m - \frac{1}{i} \partial_m \left(f(r) \frac{x_l}{r} g^{lm} \partial_k \left(g^{jk} \beta'(r) \frac{x_k}{r} \right) \right).$$

By substituting $g^{jk} = m^{jk} + (g^{jk} - m^{jk})$, it follows then that

$$\int \langle i[D_j g^{jk} D_k, Q]u, u \rangle dt \gtrsim 4 \int \int f(r) \beta'(r) \left(g^{jk} \frac{x_k}{r} \partial_j u \right)^2 dx dt \\ + \int \langle i[-\Delta, f(r) \frac{x_l}{r} D_l + D_l \frac{x_l}{r} f(r)]u, u \rangle dt - \|R^{-1}u\|_{LE_R}^2 - \|g - m\|_{1, > R} \|u\|_{LE^1_{>R}}^2.$$

Fixing $Q_0 = f(r) \frac{x_l}{r} D_l + D_l \frac{x_l}{r} f(r)$, a standard computation (see, e.g., [17]) gives

$$[-\Delta, Q_0] = \frac{4}{i} D_k \frac{x_k}{r} f'(r) \frac{x_j}{r} D_j + \frac{4}{i} D_l \left(\delta_{lk} - \frac{x_k x_l}{r^2} \right) \frac{f(r)}{r} \left(\delta_{jk} - \frac{x_k x_j}{r^2} \right) D_j - \frac{1}{i} \Delta \partial_k \left(\frac{x_k}{r} f(r) \right).$$

Using that f, f' , and $-\Delta \left((n-1) \frac{f(r)}{r} + f'(r) \right)$ are everywhere positive (on \mathbb{R}^3) and that $\frac{f(r)}{r}, f'(r) \approx R^{-1}$ and $-\Delta \left((n-1) \frac{f(r)}{r} + f'(r) \right) \approx R^{-3}$ when $|x| \approx R$, this shows that

$$\int \langle i[D_j g^{jk} D_k, Q]u, u \rangle dt \gtrsim \|\nabla_x u\|_{LE_{>R}}^2 + \| |x|^{-1} u \|_{LE_{>R}}^2 - \|R^{-1}u\|_{LE_R}^2 - \|g - m\|_{1, > R} \|u\|_{LE^1_{>R}}^2.$$

Integrating (3.6) over t , plugging this in, and using a Hardy inequality, it follows that

$$(3.7) \quad \|\nabla_x u\|_{LE_{>R}}^2 + \| |x|^{-1} u \|_{LE_{>R}}^2 \lesssim \|\partial u(T)\|_{L^2_{>R}}^2 + \|\partial u(0)\|_{L^2_{>R}}^2 + \|\square_g u\|_{LE^*_{>R}} \|Qu\|_{LE_{>R}} \\ + \|R^{-1}u\|_{LE_R}^2 + \|g - m\|_{1, > R} \|u\|_{LE^1_{>R}}^2.$$

We recover the time derivatives using a Lagrangian correction. To this end, we compute

$$\frac{d}{dt} 2\Im \langle (g^{00} D_t + 2g^{0j} D_j)u, \beta(r) f'(r) u \rangle = 2\Re \langle \square_g u, \beta(r) f'(r) u \rangle + 2\Im \langle (\partial_t g^{0j}) D_j u, \beta(r) f'(r) u \rangle \\ - 2\Im \langle (\partial_j g^{0j}) D_t u, \beta(r) f'(r) u \rangle - 2\langle g^{00} D_t u, \beta(r) f'(r) D_t u \rangle \\ + \langle \partial_k (g^{jk}) u, \partial_j (\beta(r) f'(r) u) \rangle + \langle g^{jk} u, \partial_k \partial_j (\beta(r) f'(r) u) \rangle - 2\Re \langle g^{jk} D_k u, \beta(r) f'(r) D_j u \rangle.$$

From this, it follows that

$$(3.8) \quad \|\partial_t u\|_{LE_{>R}}^2 \lesssim \|\partial u(T)\|_{L_{>R}^2}^2 + \|\partial u(0)\|_{L_{>R}^2}^2 + \|\square_g u\|_{LE_{>R}^*} \| |x|^{-1} u \|_{LE_{>R}} \\ + \|\nabla_x u\|_{LE_{>R}}^2 + \| |x|^{-1} u \|_{LE_{>R}}^2 + \|R^{-1} u\|_{LE_R}^2 + \|g - m\|_{1,>R} \|u\|_{LE_{>R}^1}^2.$$

The combination of (3.7) and (3.8) results in

$$\|u\|_{LE_{>R}^1}^2 \lesssim \|\partial u(T)\|_{L_{>R}^2}^2 + \|\partial u(0)\|_{L_{>R}^2}^2 + \|\square_g u\|_{LE_{>R}^*}^2 + \|R^{-1} u\|_{LE_R}^2 + \|g - m\|_{1,>R} \|u\|_{LE_{>R}^1}^2.$$

We finally notice that

$$\|\square_g u\|_{LE_{>R}^*} \lesssim \|Pu\|_{LE_{>R}^*} + \|(g - m, A, V)\|_{AF,>R} \|u\|_{LE_{>R}^1}.$$

Provided that \mathbf{c} in (1.3) is sufficiently small, which can be guaranteed by fixing R_0 sufficiently large, we may bootstrap to complete the proof. \square

4. HIGH FREQUENCY ANALYSIS

Here we prove Theorem 2.11. We begin with several simplifications.

First we argue that it suffices to prove (2.14) when the data and forcing term are supported in $\{|x| < 2R_0\}$. Indeed, letting \tilde{P} be a small AF perturbation of \square that agrees with P for $|x| > R_0$, we set v to solve $\tilde{P}v = f$ with $v[0] = u[0]$. Then, by Theorem 1.7, $\|v\|_{LE^1}$ is controlled by the right side of (2.14). Then it suffices to establish (2.14) for $u_1 = u - \beta_{>R_0} v$, which solves an equation with data and forcing term supported in $\{|x| < 2R_0\}$.

We next reduce to the case that $u[0] = 0$ and $f \in LE^*$. Indeed, suppose that we know

$$(4.1) \quad \|u\|_{LE^1[0,T]} + \|\partial u\|_{L^\infty L^2[0,T]} \lesssim \|\langle x \rangle^{-2} u\|_{LE} + \|Pu\|_{LE_{comp}^*[0,T]}$$

when $u[0] = 0$. Let w solve $Pw = g \in L^1 L_{comp}^2$ with nontrivial $w[0]$. We write $w = \sum_k w_k$ where $Pw_k = \mathbf{1}_{[k,k+1]}(t)g$ and $w_k[0] = 0$, $k > 0$ and $w_0[0] = w[0]$. We use $\sum_k \beta_{[k,k+1]}(t)w_k$, where $\beta_{[k,k+1]}(t)$ is a test function that is identically one on $[k, k+1]$, as an approximate solution. By (2.4) and Duhamel's principle,

$$\sum_k \left(\|\beta_{[k,k+1]} w_k\|_{LE^1} + \|\partial(\beta_{[k,k+1]} w_k)\|_{L^\infty L^2} \right) \lesssim \|\partial w(0)\|_{L^2} + \sum_k \int_k^{k+1} \|g(s)\|_2 ds.$$

The difference $w - \sum_k \beta_{[k,k+1]} w_k$ is estimated with (4.1). Using that g is compactly supported and finite propagation speed, one obtains

$$\|\langle x \rangle^{-2} \sum_k \beta_{[k,k+1]} w_k\|_{LE} + \|P(w - \sum_k \beta_{[k,k+1]} w_k)\|_{LE^*} \lesssim \sum_k \|\partial w_k\|_{L_{[k,k+1]}^\infty}^{L^2},$$

which can be bounded as before. So it remains to show (4.1).

Next we remark that, while the result in the theorem is primarily about the high frequencies, it does have a low frequency component. To account for that, we separate the high frequency part and shall prove the following estimate, which uses a frequency threshold $\lambda \gg 1$:

$$(4.2) \quad \|u_{\geq \lambda}\|_{LE^1_{<2R_0}} \lesssim \|f\|_{LE^*}^{\frac{1}{2}} \|u\|_{LE^1}^{\frac{1}{2}} + \lambda^{-\delta} \|u\|_{LE^1},$$

with an implicit constant which depends only on M_0, R_0, T_0 . Here the last term accounts for the low frequency errors.

We first show that this bound implies the desired estimate (4.1). Adding in the low frequencies this becomes

$$\|u\|_{LE^1_{<2R_0}} \lesssim \|f\|_{LE^*}^{\frac{1}{2}} \|u\|_{LE^1}^{\frac{1}{2}} + \lambda^{-\delta} \|u\|_{LE^1} + \lambda \|\langle x \rangle^{-2} u\|_{LE}.$$

From here we can also obtain the uniform energy piece. Indeed, using the Mean Value Theorem, there is a sequence $t_k \in [k, k+1]$ so that

$$\|\partial u(t_k)\|_{L_{<2R_0}^2} \lesssim \|u\|_{LE^1_{<2R_0}},$$

and (2.4) allows us to estimate the energy at an arbitrary time t by an element of this sequence. Next we apply (3.1) to complete the local norm above to a full local energy norm

$$\|u\|_{LE^1} + \|\partial u\|_{L^\infty L^2} \lesssim \|f\|_{LE^*}^{1/2} \|u\|_{LE^1}^{1/2} + \|f\|_{LE^*} + \lambda^{-\delta} \|u\|_{LE^1} + \lambda \|\langle x \rangle^{-2} u\|_{LE}.$$

Then (4.1) follows by choosing λ large enough (depending on the implicit constant, which in turn depends on M_0, R_0, T_0).

Now we turn our attention to the proof of (4.2). For this we will use the positive commutator method with a suitably chosen escape function. To simplify the argument without loss we take $g^{00} = 1$. Indeed, hypothesis (1) guarantees that g^{00} is uniformly bounded away from zero, and dividing through as in [19, Section 3] preserves the assumptions on the other coefficients. For convenience we will state separately the result concerning the existence of the escape function. For that we introduce the notations

$$p(t, \tau, x, \xi) = \tau^2 - 2g^{0j}\tau\xi_j - g^{ij}\xi_i\xi_j, \quad s_{skew} = \Im A_\alpha g^{\alpha 0} \tau + q \Im A_\alpha g^{\alpha k} \xi_k.$$

for the principal symbols of the self-adjoint, respectively the skew-adjoint parts of P . Then we have:

Lemma 4.1. *Under the same assumptions as in Theorem 2.11, there exists smooth real symbols $q = \tau q_0 + q_1$ and m where $q_0 \in S^0$, $q_1 \in S^1$ and $m \in S^0$ are supported in $|\xi|, |\tau| \gtrsim \lambda$, so that the following relation holds:*

$$(4.3) \quad \{p, q\} + pm + s_{skew}q \gtrsim \mathbf{1}_{|\xi|, |\tau| \geq \lambda} \langle x \rangle^{-2} (|\xi|^2 + \tau^2).$$

Proof. We factor $p(t, \tau, x, \xi)$ as

$$\tau^2 - 2g^{0j}\tau\xi_j - g^{ij}\xi_i\xi_j = (\tau - a^+(t, x, \xi))(\tau - a^-(t, x, \xi)).$$

Here a^\pm are 1-homogeneous in ξ and $a^+ > 0 > a^-$. See, e.g., [19, Section 6].

Set $p^\pm(t, \tau, x, \xi) = \tau - a^\pm(t, x, \xi)$. On each portion of the light cone $\tau = a^\pm(t, x, \xi)$, we first construct a multiplier q^\pm akin to that used in, e.g., [6], see also [26]. Precisely, we claim that we can find smooth homogeneous zero order symbols q^\pm with the property that

$$(4.4) \quad H_{p^\pm} q^\pm > c_k 2^{-k}, \quad |x| \approx 2^k.$$

To produce such symbols set χ to be a smooth, monotonically decreasing function with $\chi \equiv 1$ for $|x| \leq 1$ and $\chi \equiv 0$ for $|x| > 2$, and define $\chi_M(|x|) = \chi(|x|/M)$. Then, letting $(t_s^\pm, \tau_s^\pm, x_s^\pm, \xi_s^\pm)$ solve

$$\begin{aligned} \dot{t}_s^\pm &= p_\tau^\pm(t_s^\pm, \tau_s^\pm, x_s^\pm, \xi_s^\pm), & \dot{\tau}_s^\pm &= -p_t^\pm(t_s^\pm, \tau_s^\pm, x_s^\pm, \xi_s^\pm) \\ \dot{x}_s^\pm &= p_\xi^\pm(t_s^\pm, \tau_s^\pm, x_s^\pm, \xi_s^\pm), & \dot{\xi}_s^\pm &= -p_x^\pm(t_s^\pm, \tau_s^\pm, x_s^\pm, \xi_s^\pm) \end{aligned}$$

with $(t_0^\pm, \tau_0^\pm, x_0^\pm, \xi_0^\pm) = (t, \tau, x, \xi)$, set $q_{in}^\pm(t, \tau, x, \xi) = -\int_0^\infty \chi_R(|x_s^\pm|) ds$. By the nontrapping condition, this is bounded, and one can calculate $H_{p^\pm} q_{in}^\pm = \chi_R(|x|)$.

We shall supplement q_{in}^\pm with a multiplier that is inspired by that used to prove (3.5). See, also, [16]. Set

$$q_{out}^\pm = -(1 - \chi_R)(|x|) f(|x|) a_{\xi_j}^\pm \frac{x_j}{|x|}$$

where f satisfies $f(x) \approx 1$ for $|x| \geq R_0$ and $f'(|x|) \approx c_k 2^{-k}$ for $|x| \approx 2^k$.⁴ We can then compute

$$\begin{aligned} H_{p^\pm} q_{out}^\pm &= a_{\xi_k}^\pm \frac{x_k}{|x|} (1 - \chi_R)(|x|) f'(|x|) a_{\xi_j}^\pm \frac{x_j}{|x|} + a_{\xi_k}^\pm \left(\delta_{jk} - \frac{x_k}{|x|} \right) (1 - \chi_R)(|x|) \frac{f(|x|)}{|x|} \left(\delta_{jl} - \frac{x_l}{|x|} \right) a_{\xi_l}^\pm \\ &\quad - R^{-1} \chi'(|x|/R) a_{\xi_k}^\pm \frac{x_k}{|x|} f(|x|) a_{\xi_j}^\pm \frac{x_j}{|x|} + \mathcal{O}(\langle x \rangle |\partial g|) (1 - \chi_R)(|x|) |x|^{-1}, \end{aligned}$$

and note that the third term in the right hand side is nonnegative. Using (1.3), it follows that $H_{p^\pm} q_{out}^\pm$ is non-negative everywhere and strictly positive on $|x| \in [2R, 4R]$.

We now set

$$q^\pm = q_{out}^\pm + c \chi_{4R}(x) q_{in}^\pm.$$

Then

$$H_{p^\pm} q^\pm = H_{p^\pm} q_{out}^\pm + c \chi_R(|x|) - c a_{\xi_k}^\pm \frac{x_k}{|x|} \chi'_{4R}(x) q_{in}^\pm.$$

⁴Indeed, we may choose $f(r) = \exp(\sigma \int_0^r c(s) s^{-1} ds)$ where $c(s)$ is constructed from the sequence c_k as was done in [19, §2] and σ is a large parameter.

As χ'_{4R} is supported where $H_{p^\pm} q_{out}^\pm$ is strictly positive, by choosing c sufficiently small, we have that $H_{p^\pm} q^\pm$ is everywhere non-negative and positive on $|x| \leq 4R$. Thus (4.4) is satisfied.

Now we consider the truncation at frequency λ , which is also achieved at the level of q^\pm . Precisely, we set

$$q_{>\lambda}^\pm = e^{\sigma q^\pm} \chi(|\xi| e^{\sigma q^\pm} / \lambda) \chi(|\tau| e^{\sigma q^\pm} / \lambda)$$

where χ is a nondecreasing smooth cutoff selecting the interval $[1, \infty]$. We now can verify directly that

$$H_{p^\pm} q_{>\lambda}^\pm \gtrsim \sigma c_k 2^{-k} q_{>\lambda}^\pm, \quad |x| \approx 2^k$$

provided λ is large enough.

We now combine the symbols constructed on the individual light cones into

$$q(t, x, \xi, \tau) = (\tau - a^-) q_{>\lambda}^+ + (\tau - a^+) q_{>\lambda}^-,$$

and we compute

$$H_p q|_{\tau=a^\pm} = (a^+ - a^-)^2 H_{p^\pm} q_{>\lambda}^\pm + (a_\pm - a_\mp) q_{>\lambda}^\pm (-a_t^\mp + a_t^\pm + a_{\xi_k}^\pm a_{x_k}^\mp - a_{x_k}^\pm a_{\xi_k}^\mp).$$

By choosing σ sufficiently large, we have

$$H_p q|_{\tau=a^\pm} \geq \frac{1}{2} (a^+ - a^-)^2 \sigma c_k 2^{-k} q_{>\lambda}^\pm, \quad |x| \approx 2^k.$$

We now consider the principal contribution of the skew-symmetric portion of P , which has symbol

$$s_{skew} q = q \Im A_\alpha g^{\alpha 0} \tau + q \Im A_\alpha g^{\alpha k} \xi_k.$$

Since

$$s_{skew} q|_{\tau=a^\pm} = \pm (a^+ - a^-) q_{>\lambda}^\pm (\Im A_\alpha g^{\alpha 0} a^\pm + \Im A_\alpha g^{\alpha k} \xi_k),$$

the choice of large σ also allows us to absorb this, giving

$$(4.5) \quad H_p q|_{\tau=a^\pm} + s_{skew} q|_{\tau=a^\pm} \geq \frac{1}{4} (a^+ - a^-)^2 \sigma c_k 2^{-k} q_{>\lambda}^\pm, \quad |x| \approx 2^k.$$

At this point, we shall now modify by a multiple of p , which serves as a Lagrangian correction, so as to guarantee that it is everywhere nonnegative. Indeed, we note that

$$H_p q + s_{skew} q = A\tau^2 + B\tau + C$$

where A, B, C depend on (t, x, ξ) and $(A(a^\pm)^2 + Ba^\pm + C) > 0$. We seek to choose $m(t, x, \xi)$ so that

$$A\tau^2 + B\tau + C + m(\tau - a^+)(\tau - a^-) > 0.$$

It suffices to choose m so that $A + m > 0$ and

$$g(m) := (B - m(a^+ + a^-))^2 - 4(A + m)(C + ma^+a^-) < 0.$$

Taking the minimizing value for the quadratic g ,

$$m = \frac{B(a^+ + a^-) + 2C + 2Aa^-a^+}{(a^+ - a^-)^2},$$

gives

$$g(m) = -4(A(a^+)^2 + Ba^+ + C)(A(a^-)^2 + Ba^- + C)$$

and

$$A + m = \frac{(A(a^+)^2 + Ba^+ + C) + (A(a^-)^2 + Ba^- + C)}{(a^+ - a^-)^2},$$

which clearly satisfy the desired conditions.

It thus follows that

$$H_p q + s_{skew} q + mp \geq (A + m) \left(\tau + \frac{B - m(a^+ + a^-)}{2(A + m)} \right)^2 - \frac{g(m)}{4(A + m)},$$

which by (4.5) completes the proof. \square

We now return to the quantum side and use the lemma to complete the proof of (4.2). For that we compute

$$\begin{aligned}
& 2\Im \int_0^T \int P v \overline{(q^w - (i/2)m^w)} v \, dV + (1/2) \left\langle \left[m^w, \left(\Im A_\alpha g^{\alpha\beta} D_\beta + D_\alpha g^{\alpha\beta} \Im A_\beta \right) \right] v, v \right\rangle \\
& \quad - \left\langle i \left[\left(\Re A_\alpha g^{\alpha\beta} D_\beta + D_\alpha g^{\alpha\beta} \Re A_\beta \right), q^w \right] v, v \right\rangle \\
& \quad - (1/2) \left\langle \left\{ m^w \left(\Re A_\alpha g^{\alpha\beta} D_\beta + D_\alpha g^{\alpha\beta} \Re A_\beta \right) + \left(\Re A_\alpha g^{\alpha\beta} D_\beta + D_\alpha g^{\alpha\beta} \Re A_\beta \right) m^w \right\} v, v \right\rangle \\
& \quad = \langle i[\square_g, q^w] v, v \rangle + (1/2) \langle (m^w \square_g + \square_g m^w) v, v \rangle \\
& \quad + \frac{1}{i} \left\langle \left\{ q^w \left(\Im A_\alpha g^{\alpha\beta} D_\beta + D_\alpha g^{\alpha\beta} \Im A_\beta \right) + \left(\Im A_\alpha g^{\alpha\beta} D_\beta + D_\alpha g^{\alpha\beta} \Im A_\beta \right) q^w \right\} v, v \right\rangle.
\end{aligned}$$

Here $v = \beta(t - T + 2)u$. As $\|(1 - \beta(t - T + 2))u\|_{LE^1}$ and the error term $\langle [P, \beta(t - T + 2)]u, (q^w - (i/2)m^w)v \rangle$ are easily estimated using (2.4), it suffices to establish (4.2) for v .

The last three terms in the left side are lower order and are easily bounded by $\lambda^{-1/2}\|v\|_{LE^1}$ using the frequency localization of q, m . On the other hand, by the lemma, the principal symbol of the last three terms on the right side is everywhere nonnegative and bounded below by a multiple of $\mathbf{1}_{|\tau|, |\xi| \gtrsim \lambda}(\tau^2 + |\xi|^2)$ on $|x| \leq 4R_0$. Thus, by an application of Gårding's inequality, the right side is bounded below by $\|\partial v_{\geq \lambda}\|_{LE^{<4R_0}}^2$ modulo errors that are bounded by $\lambda^{-1/2}\|v\|_{LE^1}$, as desired.

5. CARLEMAN ESTIMATES AND THE MEDIUM FREQUENCY ANALYSIS

The medium frequency analysis for the local energy decay bounds is based on Carleman type estimates. This emulates in the dynamical setting the resolvent bounds used to prove the absence of embedded eigenvalues/resonances. For the latter, we refer the reader to [13]. We first discuss the Carleman estimates and then show how they apply for the local energy decay bounds.

5.1. Carleman estimates for wave equations. Here we consider two classes of Carleman estimates. One applies in the small asymptotically flat regime and the other applies in a compact set. The estimates we seek are of the form:

$$(5.1) \quad \|\omega_0 e^\varphi u\|_{L^2} + \|\omega_1 e^\varphi \nabla u\|_{L^2} \lesssim \|e^\varphi P u\|_{L^2}$$

for appropriate weights ω_0 and ω_1 , with the key feature that the implicit constant is independent of the (parameters entering the) weight φ . For our purposes it suffices to work with radial weights.

The first result is concerned with large r , so the asymptotic behavior of φ at infinity is important. Here it is convenient to express φ in the form

$$\varphi = \varphi(s), \quad s = \log r$$

and ask that it is slowly varying as function of s on the unit scale. To start, we consider the following set-up:

Proposition 5.1. *Suppose that P is asymptotically flat. Let φ be a convex weight which satisfies*

$$(5.2) \quad \lambda \lesssim \varphi'(s), \quad \lambda \lesssim \varphi''(s) \leq \frac{1}{2}\varphi'(s), \quad |\varphi'''(s)| + |\varphi^{(iv)}| \ll \varphi', \quad \lambda \gg 1.$$

Then for $u \in LE_0^1$ supported in $\{r > R_0\}$ we have the uniform estimate

$$(5.3) \quad \|r^{-1}(1 + \varphi'')^{\frac{1}{2}} e^\varphi (r^{-1}(1 + \varphi')u, \nabla u)\|_{L^2} + \|r^{-1}(1 + \varphi')^{\frac{1}{2}} e^\varphi \partial_t u\|_{L^2} \lesssim \|e^\varphi P u\|_{L^2}.$$

A variation on this theme is needed in order to deal with the fact that we want to bend the weight in such a way so that it is constant near infinity. That would make the weight nonconvex there, though we will stay with the monotonicity condition. The lack of convexity yields some errors in the estimate; the point is that these errors will be lower order, and absorbed into the main term provided that there is a bound from below on the allowed time frequencies. To simplify matters, in the next result we harmlessly assume that this transition occurs in a single dyadic region.

Proposition 5.2. *Suppose that P is asymptotically flat. Let $R > R_0$, and let φ be an increasing weight which is as in (5.2) for $s < \log R$ and is constant for $s > \log R + 1$. Then for $u \in LE_0^1$ supported in $\{r > R_0\}$ we have the uniform estimate*

$$(5.4) \quad \begin{aligned} & \|r^{-1}(1 + \varphi''_+)^{\frac{1}{2}} e^\varphi (\nabla u, r^{-1}(1 + \varphi')u)\|_{L^2_{<R}} + \|r^{-1}(1 + \varphi')^{\frac{1}{2}} e^\varphi \partial_t u\|_{L^2_{<R}} + R^{-\frac{1}{2}} \|e^\varphi u\|_{LE^1_{>R}} \\ & \lesssim \|e^\varphi P u\|_{L^2_{<R}} + R^{-\frac{1}{2}} \|e^\varphi P u\|_{LE^1_{>R}} + R^{-2} \|(1 + \varphi')^{\frac{3}{2}} e^\varphi u\|_{L^2_R}. \end{aligned}$$

Comparing the weight on the added error term with the weight on the $\partial_t u$ term on the left, we see that the error term is negligible provided that the lower bound τ_0 on the time frequencies satisfies

$$(5.5) \quad \tau_0 \gg \varphi'(\log R) R^{-1}.$$

This is quite similar to the proof in [13] of the result on the absence of embedded eigenvalues, minus the initial part which uses the symmetry of the elliptic operator to obtain the initial LE_0^1 decay at infinity (which is directly assumed here).

Proof of Propositions 5.1, 5.2. These are standard Carleman estimates. In both of them the magnetic and scalar potential terms are perturbative, so we can simply take

$$P = D_\alpha g^{\alpha\beta} D_\beta.$$

Conjugating out the exponential weight, we rewrite (5.3) in the form

$$(5.6) \quad \|r^{-1}(1 + \varphi'')^{\frac{1}{2}} (r^{-1}(1 + \varphi')v, \nabla v)\|_{L^2} + \|r^{-1}(1 + \varphi')^{\frac{1}{2}} \partial_t v\|_{L^2} \lesssim \|P_\varphi v\|_{L^2}$$

where $v = e^\varphi u$ and P_φ represents the conjugated operator

$$P_\varphi(x, D) = e^\varphi P(x, D) e^{-\varphi} = P(x, D + i\nabla\varphi).$$

This is split into a self-adjoint and a skew-adjoint part, which have the form

$$P_\varphi^r = D_\alpha g^{\alpha\beta} D_\beta - \varphi_i g^{ij} \varphi_j, \quad P_\varphi^i = iD_\alpha g^{\alpha j} \varphi_j + i\varphi_j g^{j\alpha} D_\alpha$$

where $\varphi_j = \partial_j \varphi$. Then we have

$$\|P_\varphi v\|_{L^2}^2 = \|P_\varphi^r v\|_{L^2}^2 + \|P_\varphi^i v\|_{L^2}^2 + \langle [P_\varphi^r, P_\varphi^i] v, v \rangle;$$

therefore our estimate is essentially a statement about the positivity of the commutator on the characteristic set of P_φ , i.e. the pseudoconvexity condition.

We begin by checking the pseudoconvexity condition at the symbol level. With $p(x, \xi) = g^{\alpha\beta} \xi_\alpha \xi_\beta$ we have

$$p_\varphi = g^{\alpha\beta} \xi_\alpha \xi_\beta - g^{ij} \partial_i \varphi \partial_j \varphi + 2ig^{\alpha j} \xi_\alpha \partial_j \varphi,$$

so the principal symbol of the commutator is

$$\{\Re p_\varphi, \Im p_\varphi\} = 2\{g^{\alpha\beta} \xi_\alpha \xi_\beta - g^{ij} \partial_i \varphi \partial_j \varphi, g^{\alpha j} \xi_\alpha \partial_j \varphi\}$$

All terms where g is differentiated are perturbative, so we write them as *err*. Then we have

$$\{\Re p_\varphi, \Im p_\varphi\} = 4\xi_\alpha g^{\alpha j} \varphi_{jk} g^{k\beta} \xi_\beta + 4\varphi_i g^{ij} \varphi_{jk} g^{kl} \varphi_l + \text{err}.$$

Replacing g on the right with the Minkowski metric m is also perturbative, so it suffices to do a complete computation in the Minkowski case. There we have

$$\Re p_\varphi = |\xi'|^2 - \xi_0^2 - r^{-2}(\varphi')^2, \quad \Im p_\varphi = 2r^{-1} \frac{x}{r} \cdot \xi' \varphi';$$

therefore

$$\begin{aligned} \{\Re p_\varphi, \Im p_\varphi\} &= 2\{|\xi'|^2 - r^{-2}(\varphi')^2, r^{-2} \varphi' x \cdot \xi'\} \\ &= 4\varphi'' r^{-4}((x \cdot \xi')^2 + (\varphi')^2) + 4r^{-2} \varphi' (|\xi'|^2 - r^{-2}(\varphi')^2) - 8r^{-4} \varphi' (x \cdot \xi')^2 \\ &= 4\varphi'' r^{-4}((x \cdot \xi')^2 + (\varphi')^2) + 4r^{-2} \varphi' \xi_0^2 + 4r^{-2} \varphi' \Re p_\varphi - 2(\varphi')^{-1} (\Im p_\varphi)^2. \end{aligned}$$

Then, choosing a function b with the property that $b \approx \frac{1}{2} r^{-2} \varphi''$, we have a positivity relation of the form

$$\{\Re p_\varphi, \Im p_\varphi\} + (b - 4r^{-2} \varphi') \Re p_\varphi + 2(\varphi')^{-1} (\Im p_\varphi)^2 \gtrsim \varphi'' r^{-2} (|\xi'|^2 + r^{-2}(\varphi')^2) + r^{-2} \varphi' \xi_0^2$$

where g and m are still perturbatively interchangeable. This leads to an inequality of the form

$$(5.7) \quad LHS(5.6) \lesssim \langle [P_\varphi^r, P_\varphi^i]v, v \rangle_{L^2} + \|(\varphi')^{-\frac{1}{2}} P_\varphi^i v\|_{L^2}^2 + \langle (b - 4r^{-2}\varphi')v, P_\varphi^r v \rangle$$

where all lower order terms in the commutator and in the integration by parts in the last term on the right are negligible. Since we have

$$|\varphi'| \gg 1, \quad |b - 4r^{-2}\varphi'|^2 \ll \varphi'' r^{-4}(\varphi')^2,$$

this implies (5.6).

We emphasize again the advantage of working with (5.7) and the preceding symbol bound, which is that they are stable with respect to small asymptotically flat perturbations of g . By contrast, such a substitution cannot be done directly in (5.6).

Similarly, the bound (5.4) can be written in terms of v and the conjugated operator P_φ ,

$$(5.8) \quad \begin{aligned} & \|r^{-1}(1 + \varphi''_+)^{\frac{1}{2}}(\nabla v, r^{-1}(1 + \varphi')v)\|_{L^2_{<R}} + \|r^{-1}(1 + \varphi')^{\frac{1}{2}}\partial_t v\|_{L^2_{<R}} + R^{-\frac{1}{2}}\|v\|_{LE^1_{>R}} \\ & \lesssim \|P_\varphi v\|_{L^2_{<R}} + R^{-\frac{1}{2}}\|P_\varphi v\|_{LE^*_{>R}} + R^{-2}\|(1 + \varphi')^{\frac{3}{2}}v\|_{L^2_R}. \end{aligned}$$

To prove this bound we divide the analysis into three overlapping regions:

- a) The region $\{r < R/2\}$, where φ'' is positive and (5.6) applies.
- b) The region $\{R/4 < r < 2R\}$, where $\varphi' \gg 1$ but φ'' is in part negative, with a bound $\varphi'' > -\varphi'/2$.
- c) The region $\{r > R\}$, where φ is constant and the bound (3.5) applies.

To glue the bounds in the three regions together one simply uses cutoff functions adapted to the first two regions. The third region already has the cutoff built in the estimate (3.5).

Thus it remains to consider the region (b). But there the same reasoning as above applies, with the only difference that the function b is now chosen so that $b \approx r^{-2}\varphi'$. \square

The second class of Carleman estimates applies in a compact set. Here our goal is to use the convexity of the weight in order to produce a bound which has small high frequency errors, and thus applies for time frequencies in a large but finite frequency range. This is done in order to have a result which does not use the nontrapping assumption. The result below is a local result; therefore the choice of coordinates is not essential. For simplicity we choose our weight as $\varphi = \varphi(r)$, though one can easily replace r by s away from zero.

Proposition 5.3. *Suppose P is hyperbolic, so that the surfaces $t = \text{const}$ are uniformly space-like and ∂_t is uniformly time-like. Let φ be a radial weight which satisfies*

$$(5.9) \quad \varphi'(0) = 0, \quad \varphi'' \approx \lambda + \sigma\varphi', \quad |\varphi'''| \lesssim \sigma^2\varphi', \quad |\varphi^{(iv)}| \lesssim \sigma^3\varphi', \quad \lambda, \sigma \gg 1.$$

Then we have the following estimate:

$$(5.10) \quad \|(\varphi'/r)^{\frac{1}{2}}e^\varphi\partial u\|_{L^2} + \|(\varphi'')^{\frac{1}{2}}e^\varphi\varphi'u\|_{L^2} + \|r^{-1}\varphi'e^\varphi u\|_{L^2} \lesssim \|e^\varphi Pu\|_{L^2} + \|(\varphi'/\langle r \rangle)^{\frac{1}{2}}e^\varphi\partial_t u\|_{L^2_{\geq 1}}.$$

Proof. The reasoning here is similar to the previous proof. We first note that the hypotheses (5.9) insure that φ is increasing. Next, we observe that the contribution of the potential V and the magnetic potential A in the above estimate is again perturbative since (5.9) gives that $\varphi'/r \gtrsim \lambda$, so for the purpose of this proof we set $P = D_\alpha g^{\alpha\beta} D_\beta$. After conjugation, the estimate (5.10) becomes

$$(5.11) \quad \|(\varphi'')^{\frac{1}{2}}\varphi'v\|_{L^2} + \|(\varphi'/r)^{\frac{1}{2}}\partial v\|_{L^2} + \|r^{-1}\varphi'v\|_{L^2} \lesssim \|P_\varphi v\|_{L^2} + \|(\varphi'/\langle r \rangle)^{\frac{1}{2}}\partial_t v\|_{L^2_{\geq 1}}.$$

At the symbol level we compute

$$\begin{aligned} \frac{1}{2}\{\Re p_\varphi, \Im p_\varphi\} &= \{g^{\alpha\beta}\xi_\alpha\xi_\beta - g^{ij}\partial_i\varphi\partial_j\varphi, g^{\alpha\beta}\xi_\alpha\xi_\beta\} \\ &= \partial_i\partial_j g^{i\alpha}g^{j\beta}\xi_\alpha\xi_\beta + \partial_i\partial_j\varphi g^{ik}g^{jl}\partial_k\varphi\partial_l\varphi + O(\varphi'/\langle r \rangle)(|\xi|^2 + (\varphi')^2) \\ &= \varphi''((\partial_i r g^{i\alpha}\xi_\alpha)^2 + |\nabla g^4(\varphi')|^2) + \varphi'\partial_i\partial_j r g^{i\alpha}g^{j\beta}\xi_\alpha\xi_\beta \\ &\quad + \varphi'\partial_i\partial_j r g^{ik}g^{jl}\partial_k\varphi\partial_l\varphi + O(\varphi'/\langle r \rangle)(|\xi|^2 + (\varphi')^2) \\ &\gtrsim \varphi''(\varphi')^2 + r^{-1}\varphi'|g^{i\alpha}\xi_\alpha|^2 + O(\varphi'/\langle r \rangle)(|\xi|^2 + (\varphi')^2). \end{aligned}$$

Now we split the analysis into two regions, which can be easily assembled together using an appropriate partition of unity.

1. *The inner region* $r \ll 1$. Since the surfaces $t = \text{const}$ are uniformly space-like while ∂_t is time-like, it follows that $g^{00} < 0$ while the quadratic form $g^{ij}\xi_i\xi_j$ is positive definite. Therefore we obtain an inequality of the form

$$|\xi|^2 \lesssim -\Re p_\varphi + c_0 \sum_i |g^{i\alpha}\xi_\alpha|^2$$

for a large universal constant c_0 . Thus for $r \ll 1$ we obtain the bound

$$\varphi''(\varphi')^2 + r^{-1}\varphi'|\xi|^2 \lesssim \{\Re p_\varphi, \Im p_\varphi\} - c_1 r^{-1}\varphi' \Re p_\varphi$$

with a small universal c_1 . Here no ξ_0^2 error term is needed on the right; this can be viewed as a consequence of the fact that no trapping can occur very close to $x = 0$. This translates into a bound of the form

$$\|(\varphi'')^{\frac{1}{2}}\varphi'v\|_{L^2}^2 + \|(\varphi'/r)^{\frac{1}{2}}\partial v\|_{L^2}^2 \lesssim \langle [P_\varphi^r, P_\varphi^i]v, v \rangle_{L^2} - c_1 \langle P_\varphi^r v, r^{-1}\varphi'v \rangle_{L^2}$$

where we remark that, by the uncertainty principle, the norm $\|r^{-1}\varphi'v\|_{L^2}^2$ can also be freely added on the left in order to compensate for the degeneracy of φ' at zero. Hence (5.11) easily follows for v supported in $\{r \ll 1\}$.

2. *The outer region* $r \gtrsim 1$. In this region the leading term $r^{-1}\varphi'|g^{i\alpha}\xi_\alpha|^2$ and the error term $O(\varphi'/\langle r \rangle)|\xi|^2$ are indistinguishable, so we simply write

$$\{\Re p_\varphi, \Im p_\varphi\} \gtrsim \varphi''(\varphi')^2 + O(\varphi'/\langle r \rangle)|\xi|^2.$$

Since the surfaces $t = \text{const}$ are uniformly space-like while ∂_t is time-like, it follows that $g^{00} < 0$ while the quadratic form $g^{ij}\xi_i\xi_j$ is positive definite. Thus the following must hold:

$$|\xi|^2 \lesssim \Re p_\phi + c_0(|\xi_0|^2 + |\varphi'|^2)$$

for a large universal constant c_0 . We are then led to a bound of the form

$$\varphi''(\varphi')^2 + \frac{\varphi'}{r}|\xi|^2 \lesssim \{\Re p_\varphi, \Im p_\varphi\} + c_1 \frac{\varphi'}{r} \Re p_\phi + c_2 \frac{\varphi'}{r} |\xi_0|^2$$

for large universal constants $1 \ll c_1 \ll c_2$. This translates into the bound

$$\|(\varphi'')^{\frac{1}{2}}\varphi'v\|_{L^2}^2 + \|(\varphi'/r)^{\frac{1}{2}}\partial v\|_{L^2}^2 \lesssim \langle [P_\varphi^r, P_\varphi^i]v, v \rangle_{L^2} + c_1 \langle P_\varphi^r v, (\varphi'/r)v \rangle_{L^2} + c_2 \|(\varphi'/r)^{\frac{1}{2}}\partial v\|_{L^2}^2$$

where all the lower order terms occurring in the integration by parts are negligible due to the conditions $\sigma, \lambda \gg 0$. Thus the bound (5.11) easily follows for functions v supported in the region $r \gtrsim 1$. \square

5.2. The medium frequency estimate. Our main result here is as follows:

Theorem 5.4 (Medium frequency LE bound). *Let P be an AF operator which satisfies the ellipticity condition (1). Then for each $\delta > 0$ there exists a bounded increasing radial weight $\varphi = \varphi(\log(1+r))$ so that the following bound holds for all $u \in LE_0^1$ with $Pu \in LE^*$:*

$$(5.12) \quad \begin{aligned} & \| (1 + \varphi_+'')^{\frac{1}{2}} e^\varphi \nabla u \|_{LE} + \| \langle r \rangle^{-1} (1 + \varphi_+'')^{\frac{1}{2}} (1 + \varphi') e^\varphi u \|_{LE} + \| (1 + \varphi')^{\frac{1}{2}} e^\varphi \partial_t u \|_{LE} \lesssim \\ & \delta (\| (1 + \varphi')^{\frac{1}{2}} e^\varphi u \|_{LE} + \| \langle r \rangle^{-1} (1 + \varphi_+'')^{\frac{1}{2}} (1 + \varphi') e^\varphi \partial_t u \|_{LE}) + \| e^\varphi Pu \|_{LE^*} \end{aligned}$$

with an implicit constant which is independent of δ .

This estimate is a Carleman estimate, which provides a quantitative version of the absence of positive eigenvalues embedded in the continuous spectrum of the operator $\square_g + V$. We carefully observe that this bound does not preclude the embedded resonances; this would require in addition some symmetry condition on P , which we do not assume here. The role of the parameter δ is to allow for an arbitrarily large range of time frequencies in the medium frequency bound.

Proof. This is done in a relatively straightforward manner using the Carleman estimates in Propositions 5.2, 5.3. Precisely, we will work with a weight φ which is consistent with Proposition 5.3 for $r < 2R_0$, and with Proposition 5.2 for $r > 2R_0$. For the former we fix the parameter σ to be a sufficiently large universal

constant, and retain the freedom to choose λ sufficiently large. The two λ 's must be comparable, so our weight will satisfy

$$\varphi'(s) \approx \min\{\lambda r, \lambda \log(r+10)\}, \quad \varphi''(s) \approx \lambda \quad 1 \lesssim s \leq \log R.$$

The choice here necessitates that $e^{-s}\varphi'(s) \rightarrow 0$ as $s \rightarrow \infty$, for which we have given one of many options that are easily compatible with (5.2). Combining the two Carleman estimates (this is easily done using cutoff functions and absorbing the errors) we obtain the bound

$$\begin{aligned} & \| (1 + \varphi_+'')^{\frac{1}{2}} e^\varphi (\nabla u, \langle r \rangle^{-1} (1 + \varphi') u) \|_{LE_{<R}} + \| (1 + \varphi')^{\frac{1}{2}} e^\varphi \partial_t u \|_{LE_{<R}} + \| e^\varphi u \|_{LE^1_{>R}} \\ & \lesssim \| e^\varphi P u \|_{LE^*_{<R}} + \| e^\varphi P u \|_{LE^*_{>R}} + \lambda^{1/2} \| e^\varphi \partial_t u \|_{L^2_{1 \lesssim \cdot < 2R_0}} + R^{-3/2} \| (1 + \varphi')^{\frac{3}{2}} e^\varphi u \|_{L^2_R} \end{aligned}$$

where the last two terms on the right are the error terms in the two Propositions. This implies the bound (5.12) provided that we choose λ large enough for the first error term and R large enough for the second (using that $R^{-1}\varphi'(\log R) \rightarrow 0$).

If we want to be more accurate, suppose that we restrict the above estimate to functions with time frequencies in the band $\tau_1 \leq |\xi_0| \leq \tau_2$. Then we can absorb the first error provided that $\tau_2 \ll \lambda$, and the second error provided that $\varphi'(\log R)R^{-1} \ll \tau_1$. \square

6. LOW FREQUENCY ANALYSIS

This section is devoted to a more detailed analysis of the low frequency regime. One of the main goals will be to prove the following result, which shows that the zero nonresonance condition (2.13) provides full information in a neighbourhood of time frequency zero.

Theorem 6.1 (Low frequency LE bound). *Let P be an AF operator which satisfies the ellipticity condition (1) and the zero nonresonance condition (2.13) uniformly in time. Then a modified form of stationary local energy decay holds,*

$$(6.1) \quad \| u \|_{LE^1} \lesssim \| \partial_t u \|_{LE^1_{comp}} + \| P u \|_{LE^*}$$

for all compactly supported u .

Comparing this with the earlier stationary local energy decay condition (1.7), we remark that the above bound is stronger in that only a local norm of $\partial_t u$ is used on the right, but also weaker in that derivatives of $\partial_t u$ are also used. However, the latter fact is harmless, as the above bound is only used at very low time frequencies. This theorem yields directly just one of the relations in Proposition 2.10, precisely the one we need for the proof of the two point local energy decay bound in Theorem 2.15. However, the results we establish along the way also suffice for the full proof of Proposition 2.10. We discuss this at the end of the section.

The first step is to show that, in the context of the present work, the zero resolvent bound (2.13) is stable with respect to small perturbations of the operator P and can be deduced from a more standard non-quantitative form. It also holds uniformly when V is sufficiently non-negative. This is all contained in:

Lemma 6.2. *Let P be stationary, asymptotically flat, so that the time slices $t = \text{const}$ are uniformly space-like and ∂_t is uniformly time-like. Then one has the following properties related to the zero resolvent bound (2.13):*

- i) (Stability) *The zero resolvent bound (2.13) is stable with respect to sufficiently small stationary AF perturbations of g, A, V .*
- ii) (Non-quantitative version) *Suppose that there exists no distributional solutions to $P_0 u_0 = 0$ with $u_0 \in H^1(\mathbb{R}^3)$. Then there exists a $K_0 = K_0(t_0)$ such that estimate (2.13) holds.*
- iii) (Bound for sufficiently non-negative V) *Suppose that P is symmetric and that:*

$$(6.2) \quad - \int_{\mathbb{R}^3} \mathcal{Q}(u, u) dx \leq \lambda_0 \int_{\mathbb{R}^3} |D_A u|^2 dx, \quad \text{where } \mathcal{Q}(u, u) = 2A_0 g^{0i} \Re(u \overline{(D_i + A_i)u}) + (g^{00} A_0^2 + V)|u|^2$$

and $|D_A u|^2 = g^{jk}(\partial_j + iA_j)u \cdot \overline{(\partial_k + iA_k)u}$. Then, if $\lambda_0 < 1$, the estimate (2.13) holds with $K_0 = O((1 - \lambda_0)^{-1})$. In particular one has (2.13) for symmetric AF magnetic wave equations $\square_{g,A} + W$ if $A_0 = 0$ and $W \geq 0$.

Remark 6.3. Note that in case ii) above we do not gain quantitative control on K_0 . Therefore the assumption that there exist no H^1 solutions to $Qu_0 = 0$ becomes less useful in the case when g, V are non-stationary, and one must instead rely directly on (2.13).

Proof of Lemma 6.2. The three parts are proved separately.

i) Using the Hardy estimate

$$(6.3) \quad \| |x|^{-1} \phi \|_{L^2(\mathbb{R}^3)} \leq 2 \| \nabla \phi \|_{L^2(\mathbb{R}^3)},$$

and its dual we can enhance (2.13) to:

$$(6.4) \quad \| (\nabla u, \langle x \rangle^{-1} u) \|_{L^2(\mathbb{R}^3)} \lesssim K_0 \| P_0(x, D) u \|_{\dot{H}^{-1} + \langle r \rangle^{-1} L^2(\mathbb{R}^3)}.$$

Now its stability with respect to small AF perturbations is straightforward.

ii) The proof of follows from a standard compactness argument. First assume there exists a sequence $u_n \in \dot{H}^1$ normalized so $\| u_n \|_{\dot{H}^1} = 1$ and $\| P_0 u_n \|_{\dot{H}^{-1}} \rightarrow 0$. Then WLOG we can assume $u_n \rightharpoonup u_0$ where $\| u_0 \|_{\dot{H}^1} \leq 1$ and $P_0 u_0 = 0$ in the sense of distributions. By the ellipticity of P_0 , this convergence must be strong on compact sets.

There are now two scenarios: either $u_0 \neq 0$ or $u_n \rightarrow 0$ strongly in $H^1(|x| \leq R)$ for all $R > 0$. In the first case we have reached a contradiction. The latter case is ruled out since for $\{r > R_0\}$ the operator P_0 is a small perturbation of Δ ; therefore we have the truncated bound

$$\| u_n \|_{\dot{H}^1(\{r > R_0\})} \lesssim \| u_n \|_{H^1(\{r \approx R_0\})} + \| P_0 u_n \|_{\dot{H}^{-1}}.$$

iii) Integrating by parts we obtain:

$$(6.5) \quad \int_{\mathbb{R}^3} (|D_A u|^2 + \mathcal{Q}(u, u)) dx = \int_{\mathbb{R}^3} P_0 u \cdot \bar{u} dx.$$

Thus Kato's inequality $|\nabla_x u|^2 \lesssim |D_A u|^2$ and the condition $\lambda_0 < 1$ on line (6.2) give:

$$\| u \|_{\dot{H}^1(\mathbb{R}^3)}^2 \lesssim (1 - \lambda_0)^{-1} \| P_0 u \|_{\dot{H}^{-1}(\mathbb{R}^3)} \| u \|_{\dot{H}^1(\mathbb{R}^3)}.$$

On the other hand going back to (6.5) and instead using Hardy's estimate (6.3) to bound the magnetic terms gives

$$(6.6) \quad \int_{\mathbb{R}^3} |Du|^2 dx \lesssim \| u \|_{\dot{H}^1}^2 + \| u \|_{\dot{H}^1} \| u \|_{\dot{H}^1} + \| P_0 u \|_{\dot{H}^{-1}(\mathbb{R}^3)} \| u \|_{\dot{H}^1(\mathbb{R}^3)}.$$

The desired conclusion follows by combining the last two lines. \square

Next we establish the following simple lemma for the Euclidean Laplacian $\Delta = \sum \partial_k^2$, which shows that it satisfies a family of weighted local energy type bounds:

Lemma 6.4. *For $s = 0, 1, 2$ one has the estimates*

$$(6.7) \quad \begin{aligned} \Delta^{-1} : \ell^1 L^{2, \frac{1}{2}} &\longrightarrow \ell^\infty L^{2, -\frac{3}{2}}, \\ \nabla^s \Delta^{-1} : \ell^1 L^{2, \frac{1}{2}} &\longrightarrow \ell^1 L^{2, -\frac{3}{2} + s}, \quad s = 1, 2, \\ \nabla^s \Delta^{-1} : \ell^1 L^{2, \frac{3}{2}} &\longrightarrow \ell^\infty L^{2, -\frac{1}{2} + s}, \quad s = 0, 1, 2. \end{aligned}$$

Proof. By scaling it suffices to solve the equation $\Delta u = f$ with $f \in L^2$ supported in $\{r \approx 1\}$. By elliptic regularity we have $u \in H^2(\{r \approx 1\})$. Inside a small ball centered at zero we have uniform bounds for u and its derivatives, which suffice for all bounds above. Outside a large ball centered at zero we have pointwise decay estimates inherited from the fundamental solution of Δ . Precisely, $\nabla^s u$ decays like r^{-1-s} . This again suffices for all bounds in the lemma. \square

The following step is to explore some equivalent formulations of the zero resolvent estimate (2.13), where different weights are used at infinity or where the bound is restricted to a large ball.

Lemma 6.5 (Zero frequency LE bound). *Assume that P_0 is asymptotically flat and that the zero resolvent estimate (2.13) holds. Then*

a) For each $u \in \mathcal{LE}_0^1$ we have

$$(6.8) \quad \|\langle r \rangle u\|_{\mathcal{LE}^1} \lesssim K_0 \|\langle r \rangle P_0 u\|_{\mathcal{LE}^*}.$$

b) For each $u \in \mathcal{LE}_0^1$ we have

$$(6.9) \quad K_0^{-1} \|\langle r \rangle u\|_{\mathcal{LE}_{<K_0}^1} + \|u\|_{\mathcal{LE}_{>K_0}^1} + \|\langle r \rangle^{-1} \nabla u\|_{\mathcal{LE}_{>K_0}^*} \lesssim \|P_0 u\|_{\mathcal{LE}^*}.$$

c) For each $u \in \mathcal{LE}_{loc}^1$ and $R_1 \gg K_0, R_0$ we have

$$(6.10) \quad K_0^{-1} \|\langle r \rangle u\|_{\mathcal{LE}_{<K_0}^1} + \|u\|_{\mathcal{LE}_{R_1>\cdot>K_0}^1} + \|\langle r \rangle^{-1} \nabla u\|_{\mathcal{LE}_{R_1>\cdot>K_0}^*} \lesssim \|P_0 u\|_{\mathcal{LE}_{<R_1}^*} + \|r^{-1} \nabla(ru)\|_{\mathcal{LE}_{R_1}}.$$

In the estimate in part (c), the last term on the right plays the role of a boundary term where $\{r \approx R_1\}$. Its form is based on the fundamental solution r^{-1} for Δ , which captures well the behavior at infinity of u .

Proof of Proposition 6.5. The proof of the proposition is perturbative off the Laplacian, and the bound (2.13) is used as a black box in a compact set. We begin with the bounds (6.8) and (6.9). The above lemma shows that the two bounds hold with $K_0 = 1$ if $P_0 = \Delta$. Further, using the $s = 2$ cases of (6.7) and absorbing the errors directly, they hold for small asymptotically flat perturbations of Δ . To use this fact, we consider a small perturbation \tilde{P}_0 of Δ which agrees with P_0 for $r > R_0$. Then the function $\tilde{P}_0^{-1} P_0 u$ satisfies the bounds (6.8) and (6.9) with $K_0 = 1$. It remains to estimate the difference

$$\tilde{u} = u - \tilde{P}_0^{-1} P_0 u,$$

which satisfies an equation with right hand side

$$P_0 \tilde{u} = -(P_0 - \tilde{P}_0) \tilde{P}_0^{-1} P_0 u.$$

This is supported in $\{r < R_0\}$ and satisfies

$$\|P_0 \tilde{u}\|_{L^2} \lesssim \|P_0 u\|_{\mathcal{LE}^*}.$$

Then by (2.13) we obtain

$$\|\nabla \tilde{u}\|_{L^2} \lesssim K_0 \|P_0 u\|_{\mathcal{LE}^*},$$

which suffices for $r \lesssim R_0$. For larger r we truncate to see that

$$\|\tilde{P}_0(\chi_{>R_0} \tilde{u})\|_{L^2} = \|P_0(\chi_{>R_0} \tilde{u})\|_{L^2} \lesssim K_0 \|P_0 u\|_{\mathcal{LE}^*}.$$

Thus applying the bound (6.8) for \tilde{P}_0 we obtain

$$\|\langle r \rangle \tilde{u}\|_{\mathcal{LE}^1} \lesssim K_0 \|P_0 u\|_{\mathcal{LE}^*},$$

which suffices for both (6.8) and (6.9).

We now turn our attention to the truncated bound (6.10). We shall apply the bound (6.9) to the function

$$u_1 = \chi_{<R_1} u + r^{-1} \chi_{>R_1} (ru)_{R_1}$$

where the latter factor $(ru)_R$ stands for the average of ru in the R annulus. By the Poincaré inequality we have the gluing relation

$$\|\nabla(u - r^{-1}(ru)_R)\|_{L_R^2} + R^{-1} \|u - r^{-1}(ru)_R\|_{L_R^2} \lesssim \|r^{-1} \nabla(ru)\|_{L_R^2}.$$

Then we compute

$$P_0 u_1 = \chi_{<R_1} P_0 u + [P_0, \chi_{<R_1}](u - r^{-1}(ru)_{R_1}) + \chi_{>R_1} (ru)_{R_1} (P_0 - \Delta) r^{-1},$$

and using the previous bound we estimate

$$\|P_0 u_1\|_{\mathcal{LE}^*} \lesssim \|\chi_{<R_1} P_0 u\|_{\mathcal{LE}^*} + R_1^{-\frac{1}{2}} \|r^{-1} \nabla(ru)\|_{L_{R_1}^2} + c R_1^{-1} |(ru)_{R_1}|,$$

while

$$\begin{aligned} K_0^{-1} \|\langle r \rangle u_1\|_{\mathcal{LE}_{<K_0}^1} + \|u_1\|_{\mathcal{LE}_{>K_0}^1} + \|\langle r \rangle^{-1} \nabla u_1\|_{\mathcal{LE}_{>K_0}^*} \\ \approx K_0^{-1} \|\langle r \rangle u\|_{\mathcal{LE}_{<K_0}^1} + \|u\|_{\mathcal{LE}_{R_1>\cdot>K_0}^1} + \|\langle r \rangle^{-1} \nabla u\|_{\mathcal{LE}_{R_1>\cdot>K_0}^*} + R_1^{-1} |(ru)_{R_1}|. \end{aligned}$$

Thus, applying (6.9) and using $\mathbf{c} \ll 1$ we obtain the desired bound (6.10). \square

Proof of Theorem 6.1. Our main estimate (6.1) will follow from a combination of the outer estimate (3.2) and the zero frequency LE bounds in Lemma 6.5. For that we choose parameters $R_0 \ll R_1$, as well as an intermediate $R_0 < R_2 < R_1$. Applying (3.2) with $R = R_2$ we have

$$(6.11) \quad \|u\|_{LE^1_{>R_2}} \lesssim \|\partial u\|_{LE_{R_2}} + \|Pu\|_{LE^*_{>R_2}}.$$

On the other hand, integrating the bound (6.10) in time in L^2 gives

$$(6.12) \quad K_0^{-1} \|\langle r \rangle u\|_{LE^1_{<K_0}} + \|u\|_{LE^1_{R_1 > \cdot > K_0}} \lesssim \|P_0 u\|_{LE^*_{<R_1}} + \|u\|_{LE^1_{R_1}}.$$

Further, by the pigeonhole principle, we can choose R_2 so that

$$(6.13) \quad \|\nabla u\|_{LE_{R_2}} \lesssim \|P_0 u\|_{LE^*_{<R_1}} + [\log(R_1/K_0)]^{-1} \|u\|_{LE^1_{R_1}}.$$

Combining the three we obtain

$$K_0^{-1} \|\langle r \rangle u\|_{LE^1_{<K_0}} + \|u\|_{LE^1_{>K_0}} \lesssim \|Pu\|_{LE^*_{>R_2}} + \|P_0 u\|_{LE^*_{<R_1}} + \|\partial_t u\|_{LE_{R_2}}.$$

Then the desired bound (6.1) is obtained by replacing P_0 above with P , with errors involving only time derivatives of u . \square

Proof of Proposition 2.10. We begin with the observation that the stability property with respect to small AF perturbations of P was proved based on property (i) in Lemma 6.2. We now consider the equivalence of properties (a)-(e), as follows:

- (a) \implies (c) was proved in Lemma 6.2 (ii).
- (c) \implies (b) was proved in Lemma 6.5 (b).
- (b) \implies (a) is trivial.
- (c) \implies (e) is the main result of Theorem 6.1.
- (e) \implies (a) is proved below.
- (c) \implies (d) repeats the proof of Theorem 6.1 but for the resolvent equation.
- (d) \implies (b) is trivial.

It remains to show (e) \implies (a). Suppose $P_0 u_0 = 0$ for some $u_0 \in \mathcal{LE}_0^1$. Then u_0 decays at infinity in an averaged sense, so by elliptic estimates near infinity we must have $u_0 \in \dot{H}^1$.

On the other hand, in (6.1) one can add initial and final data using the extension argument in Section 7.1 to obtain the bound

$$\|u\|_{LE^1} \lesssim \|\partial_t u\|_{LE_{comp}} + \|Pu\|_{LE^*} + \|\partial u(0)\|_{L^2} + \|\partial u(T)\|_{L^2}.$$

Applying this to the time independent function u_0 we obtain

$$T^{\frac{1}{2}} \|u_0\|_{\mathcal{LE}^1} \lesssim \|u_0\|_{\dot{H}^1},$$

and the desired conclusion $u_0 = 0$ follows by letting $T \rightarrow \infty$. \square

7. TWO POINT LOCAL ENERGY DECAY AND ENERGY GROWTH

The goal of this section is to prove Theorems 2.15 and 2.16.

7.1. An extension argument. Our first goal is to reduce Theorem 2.15 to a simpler version of it, without boundary terms at the initial and final time:

Theorem 7.1 (Unconditional local energy decay). *Assume that P is nontrapping, asymptotically flat, and satisfies the zero nonresonance condition (2.13). Assume in addition that P is ϵ -almost stationary, where ϵ is small enough*

$$\epsilon \ll_{R_0, M_0, T_0, K_0} 1.$$

Then for all u in the Schwartz space $\mathcal{S}(\mathbb{R}^{3+1})$, we have

$$(7.1) \quad \|u\|_{LE^1} \lesssim \|Pu\|_{LE^*}.$$

The advantage to not having initial and final boundary terms in (7.1) is that it is easier to frequency localize it with respect to time.

Proof that Theorem 7.1 implies Theorem 2.15. We begin with some straightforward simplifications. First, by direct energy estimates we obtain the bound

$$\|\partial u\|_{L^\infty L^2} \lesssim \|\partial u(0)\|_{L^2} + \|u\|_{LE^1} + \|Pu\|_{L^1 L^2 + LE^*}.$$

Thus it suffices to show that

$$(7.2) \quad \|u\|_{LE^1[0,T]} \lesssim \|\partial u(0)\|_{L^2} + \|\partial u(T)\|_{L^2} + \|Pu\|_{L^1 L^2 + LE^*}.$$

Secondly, also by straightforward energy estimates and finite speed of propagation, we can assume that u is supported in $\{r \lesssim T\}$.

Denote $Pu = f$. It suffices to construct a function v that matches the Cauchy data of u at times 0 and T and satisfies the bound

$$(7.3) \quad \|v\|_{LE^1} + \|Pv - f\|_{LE^*} \lesssim \|\partial u(0)\|_{L^2} + \|\partial u(T)\|_{L^2} + \|f\|_{L^1 L^2 + LE^*}.$$

Then the desired estimate (7.2) is obtained by applying the bound (7.1) to $u - v$.

To produce v , we use a partition of unity to localize the problem to three regions for the support of $u[0]$, $u[T]$ and f .

The inner region $D_{in} = \{r < 4R_0\}$. We split this region into unit time intervals D_{in}^j using a time partition of unity $1 = \sum \chi_j^2$. In each of these time intervals we use local theory and standard energy estimates to solve the equation $Pv^j = \chi^j(t)f$, making sure that we match the initial data for u , respectively the final data for u , in the first, respectively the last of these intervals. Then we reassemble these localized solutions by setting $v = \sum \chi^j v_j$.

The outer region $D_{out} = \{r > 2R_0\}$. Here we will produce v localized to $\{r > R_0\}$, so that the inner part of the operator P is not involved at all. Then, without any restriction in generality we can assume that P is a small AF perturbation of \square .

By time reversal symmetry we can assume that $u[T] = 0$, and f is supported in $\{t < 3T/4\}$. Then we use Theorem 1.7 to solve the problem

$$\tilde{P}w = f, \quad w[0] = u[0],$$

where \tilde{P} is a small AF perturbation of \square that coincides with P on $\{r > R_0\}$. The function w satisfies a good bound,

$$\|w\|_{LE^1} + \|\partial w\|_{L^\infty L^2} \lesssim \|\partial u(0)\|_{L^2} + \|f\|_{L^1 L^2 + LE^*},$$

but it is not supported in $\{r > R_0, t < T\}$. We remedy this with suitable cutoff functions, and set

$$v = \chi_{>R_0}(r)\chi_{<T}(t)w.$$

This still satisfies the LE^1 bound, but we need to estimate the truncation error,

$$Pv - f = [P, \chi_{>R_0}(r)]\chi_{<T}(t)w + \chi_{>R_0}(r)[P, \chi_{<T}(t)]w.$$

The first term is supported in $\{R_0 \leq r \leq 2R_0\}$ and is easily estimated in LE^* in terms of the LE^1 norm of w . The second is supported in $\{R_0 < r < CT, 3T/4 \leq t < T\}$ and satisfies the bound

$$\|[P, \chi_{<T}(t)]w\|_{L^\infty L^2} \lesssim T^{-1}(\|\partial w\|_{L^\infty L^2} + \|r^{-1}w\|_{L^\infty L^2}) \lesssim T^{-1}\|\partial w\|_{L^\infty L^2},$$

where the Hardy inequality was used at the last step. This bound is easily converted to an LE^* bound. \square

7.2. Time frequency localization. Here we prove Theorem 7.1 through three building blocks, which deal with low, medium, respectively high frequencies. Precisely, these three building blocks are the estimates (6.1), (5.12) and (2.14). The idea is to apply each of the three estimates to u truncated to appropriate frequency ranges and sum up.

To be precise, the error in (6.1) can be absorbed on the left provided that the u is time frequency localized in $\{|\xi_0| \leq 2\tau_0\}$ for a small enough τ_0 ,

$$\tau_0 \ll_{M_0, R_0, K_0} 1.$$

Similarly, the error in (2.14) can be absorbed on the left provided that the u is time frequency localized in $\{|\xi_0| > \tau_1\}$ for a large enough τ_1 ,

$$\frac{1}{\tau_1} \ll_{M_0, R_0, T_0} 1.$$

Finally, by choosing δ small enough, $\delta \ll_{M_0, R_0, K_0, T_0} 1$, we insure that the error in (5.12) can be absorbed on the left provided that the u is time frequency localized in $\{\tau_0 < |\xi_0| < 2\tau_1\}$. The implicit constant we obtain there depends on the range of φ , which in turn depends on δ .

The above reasoning shows that we have the desired bound (2.16) provided that u is time frequency localized in one of the three ranges above. For a general u , we simply use time frequency multipliers to divide it into suitable parts. To conclude the argument we need to be able to absorb the frequency truncation errors. For that, we should estimate the commutator of P with multipliers $Q_{\leq 1} = Q_{\leq 1}(D_t)$ whose C_0^∞ symbols are equal to 1 in a neighbourhood of zero. Precisely, it suffices to prove the following:

Proposition 7.2. *Assume that P is asymptotically flat and ϵ - slowly varying. Then for Q as above we have the estimate*

$$(7.4) \quad \|[P, Q_{\leq 1}]u\|_{LE^*} \lesssim \epsilon(\|u\|_{LE^1} + \|Pu\|_{LE^*}).$$

Proof. As the commutator will also involve terms containing two spatial derivatives of u , we first need to estimate those. We claim that the following elliptic bound holds:

$$(7.5) \quad \|\partial_x^2 u\|_{\partial_t LE + \langle r \rangle^{-1} LE} \lesssim \|u\|_{LE^1} + \|Pu\|_{LE^*}.$$

To see this we first observe that we can freely truncate u first to a spatial dyadic region $\{|x| \approx 2^k\}$, and then to a time interval of similar size 2^k . Rescaling, we can reduce the problem to the case when u is supported in a unit ball B

$$\|\partial_x^2 u\|_{\partial_t L^2 + L^2} \lesssim \|u\|_{H^1} + \|Pu\|_{L^2}.$$

Next we write the equation for u in the elliptic form

$$-(\partial_i g^{ij} \partial_j + \partial_t^2)u = \partial_t f_1 + f_2, \quad f_1 = 2g^{0j} \partial_j u + (g^{00} - 1)\partial_t u$$

where the terms f_1 and f_2 have the regularity

$$\|f_1\|_{L^2} + \|f_2\|_{L^2} \lesssim \|u\|_{H^1} + \|Pu\|_{L^2}.$$

Thus it remains to show that for u supported in a unit ball B we have the bound

$$\|\partial_x^2 u\|_{\partial_t L^2 + L^2} \lesssim \|u\|_{H^1} + \|Lu\|_{\partial_t L^2 + L^2}, \quad L = -(\partial_i g^{ij} \partial_j + \partial_t^2).$$

Here, after rescaling, we know that the coefficients of L are uniformly of class $C^{1,1}$. The operator L is elliptic and coercive; therefore its inverse is well defined and has the same mapping properties as $(-\Delta)^{-1}$. Denote $v = L^{-1}f_1 \in H_{x,t}^2$, where we use, say, Dirichlet boundary conditions on a larger ball $4B$. Then it suffices to estimate the difference $w = u - \partial_t \chi_{2B} v$. This is still compactly supported, and solves

$$Lw = f_2 + [L, \partial_t \chi_{2B}]v \in L^2.$$

Thus by elliptic regularity we have $w \in H^2$, and the proof of (7.5) is concluded.

In view of (7.5), the estimate (7.4) reduces to

$$(7.6) \quad \|[P, Q_{\leq 1}]u\|_{LE^*} \lesssim \epsilon(\|u\|_{LE^1} + \|\partial_x^2 u\|_{\partial_t LE + \langle r \rangle^{-1} LE}).$$

We now split the coefficients of P into low and high time frequencies

$$P = P_{\ll 1} + P_{\gtrsim 1}.$$

For the low frequency part, all the interactions in the commutator are localized at fixed time frequency. We represent the commutator in the form

$$[P_{\ll 1}, Q_{\leq 1}]u(t) = \int a(t_1, t_2) P_{t, \ll 1}(t + t_1) Q_1 u(t + t_2) dt_1 dt_2$$

where P_t is obtained from P by differentiating its coefficients in time, Q_1 is a multiplier localized at time frequency 1 and a is a Schwartz function. Since the bound we use for u and the assumption on P are both invariant with respect to time translations, it suffices to show that

$$\|P_{t, \ll 1} Q_1 u\|_{LE^*} \lesssim \epsilon(\|u\|_{LE^1} + \|\partial_x^2 u\|_{\partial_t LE + \langle r \rangle^{-1} LE}).$$

Indeed, we have

$$\|P_{t, \ll 1} Q_1 u\|_{LE^*} \lesssim (\| \langle r \rangle g_{t, \ll 1} \|_1 + \| \langle r \rangle A_{t, \ll 1} \|_1 + \| \langle r \rangle V_{t, \ll 1} \|_0) (\|Q_1 u\|_{LE} + \|Q_1 \partial_x u\|_{LE} + \|Q_1 \partial_x^2 u\|_{LE}),$$

which implies the previous bound.

For the high frequency part there is no commutator to worry about, so we have two terms to estimate. For the first we use the time frequency localization directly to write

$$\|P_{\gtrsim 1} Q_{\leq 1} u\|_{LE^*} \lesssim (\|g_{\gtrsim 1}\|_1 + \| \langle r \rangle A_{\gtrsim 1} \|_1 + \| \langle r \rangle^2 V_{\gtrsim 1} \|_0) (\|Q_{\leq 1} u\|_{LE^1} + \|Q_{\leq 1} \partial_x^2 u\|_{LE})$$

which suffices. For the second we represent

$$\partial_x^2 u = \partial_t f_1 + f_2, \quad f_1 \in LE, \quad f_2 \in \langle r \rangle^{-1} LE,$$

and commute out the time derivative in the second order spatial terms in P ,

$$g \partial_x^2 u = \partial_t (g f_1) - g_t f_1 + g f_2.$$

This allows us to estimate

$$\|Q_{\leq 1} P_{\gtrsim 1} u\|_{LE^*} \lesssim (\|g_{\gtrsim 1}\|_2 + \| \langle r \rangle A_{\gtrsim 1} \|_1 + \| \langle r \rangle^2 V_{\gtrsim 1} \|_0) (\|u\|_{LE^1} + \|\partial_x^2 u\|_{\partial_t LE + \langle r \rangle^{-1} LE}),$$

which again suffices. \square

7.3. The exponential energy dichotomy. Here we prove Theorem 2.16. The idea is to combine the two point local energy decay bound (2.16) with an energy relation for the inhomogeneous equation

$$(7.7) \quad Pu = f \in L^1 L^2, \quad u[0] = (u_0, u_1) \in \dot{H}^1 \times L^2.$$

The energy relation is closely connected with both the symmetry of P and the stationarity of P . If both these properties hold, then we have a conserved energy E , but which might not be positive definite. However, it is positive definite outside a compact set, so for solutions to the homogeneous equation, the relation $E(u(0)) = E(u(T))$ implies that

$$(7.8) \quad \|\partial u\|_{L^\infty L^2(0, T)} \lesssim \|\partial u(0)\|_{L^2} + \|\partial^{\leq 1} u(T)\|_{L_{comp}^2}.$$

Here $\partial^{\leq 1} u = \sum_{|\alpha| \leq 1} \partial^\alpha u$. The general case uses the following slight extension of this:

Lemma 7.3. *Assume that the asymptotically flat operator P is ϵ -almost symmetric and ϵ -slowly varying. Then we have an energy relation of the form*

$$(7.9) \quad \|\partial u\|_{L^\infty L^2(0, T)} \lesssim \|\partial u(0)\|_{L^2} + \|\partial^{\leq 1} u(T)\|_{L_{comp}^2} + \epsilon \|u\|_{LE^1} + \epsilon^{-1} \|f\|_{L^1 L^2 + LE^*}.$$

Proof. In a first step, by solving an exterior problem in the small asymptotically flat region we reduce the problem to the case when $f \in LE_{comp}^*$.

Assume first that P is symmetric. Then we can define the energy as before, except that it is no longer conserved. Instead, the regularity of the coefficients insures that the time derivative of the energy can be estimated in terms of the local energy.

If P is not symmetric, then the same applies for the energy associated to its symmetrization. Thus we can write the energy relation

$$E(u(T)) = E(u(0)) + O(\epsilon) \|u\|_{LE^1[0, T]}^2 + \int_0^T \int |f u_t| dx dt.$$

Since $f \in LE^*$, we can use Cauchy-Schwarz for the last term and the conclusion follows. \square

For the remainder of this section we use (7.9) and (2.16) to prove the conclusion of Theorem 2.16. Denote by

$$D = \|\partial u(0)\|_{L^2}^2 + \|f\|_{L^1 L^2 + LE^*}^2, \quad E(t) = \|\partial u(t)\|_{L^2}^2, \quad E_{comp}(t) = \|\partial^{\leq 1} u(t)\|_{L_{comp}^2}^2$$

the size of the data, respectively the solution size and the compact error.

Combining (7.9) and (2.16) we obtain

$$(7.10) \quad E(T) \lesssim D + E_{comp}(T)$$

while by (2.16),

$$(7.11) \quad \int_0^T E_{comp}(t) dt \lesssim D + E(T).$$

Thus we get

$$(7.12) \quad \int_0^T E(t) dt \lesssim TD + E(T).$$

The function

$$m(T) = \int_0^T E(t) dt$$

solves the differential inequality

$$m'(T) \geq cm(T) - CTD.$$

Once we pass the threshold $m(T) = 2c^{-1}CTD$, we cannot cross it back; so $m(T)$ must grow at a fixed exponential rate. Otherwise we must have

$$\int_0^T E(t) dt \lesssim TD.$$

Thus on a sequence $T_n \rightarrow \infty$ we have

$$E(T_n) \lesssim D.$$

Then from (2.16) we get uniform energy boundedness and local energy decay.

8. SPECTRAL THEORY IN THE TIME INDEPENDENT CASE

8.1. Local energy decay vs. resolvent bounds. Here we prove Proposition 2.3, which establishes the equivalence between local energy decay and resolvent bounds. For convenience we restate the two estimates. The local energy decay bound has the form

$$(8.1) \quad \|u\|_{LE^1[0,T]} + \|\partial u\|_{L^\infty L^2[0,T]} \lesssim \|\partial u(0)\|_{L^2} + \|Pu\|_{LE^*+L^1 L^2[0,T]},$$

while the resolvent local energy bound is

$$(8.2) \quad \|R_\omega\|_{\mathcal{LE}^* \rightarrow \mathcal{LE}_\omega^1} \lesssim 1, \quad \Im \omega < 0.$$

A. Local energy decay implies LE resolvent bounds. A direct consequence of the local energy decay estimate (8.1) is the uniform energy estimate

$$Pu = 0 \implies \|\partial u\|_{L^\infty L^2[0,\infty)} \lesssim \|\partial u(0)\|_{L^2}.$$

This in turn shows, via the formula (2.3), that the resolvent R_ω is defined and holomorphic in the lower half-space, and the resolvent bound (2.7) holds. We remark that for $-\Im \omega \gtrsim 1$, the bound (8.2) follows from (2.7). Thus we are left with the case $-1 < \Im \omega < 0$.

Let $u \in \dot{H}^1$ so that $P_\omega u = f$. Then we can produce an exponentially growing solution for the inhomogeneous P equation by setting

$$v = e^{i\omega t} u, \quad g = e^{i\omega t} f, \quad Pv = g.$$

We apply the local energy bound (8.1) to v on the time interval $[-T, 0]$ and then let $T \rightarrow \infty$. This yields the bound

$$(8.3) \quad \|u\|_{\mathcal{LE}_\omega^1} + |\Im \omega|^{\frac{1}{2}} \|u\|_{\dot{H}_\omega^1} \lesssim \|f\|_{\mathcal{LE}^* + |\Im \omega|^{\frac{1}{2}} L^2},$$

which implies (8.2).

B. LE resolvent bounds imply local energy decay. We first observe that the resolvent LE bounds (8.2) imply the resolvent energy bounds (2.7). Indeed, if resolvent LE bounds hold, then in particular there are no eigenvalues in the lower half-plane. Hence, by Fredholm theory, the resolvent is holomorphic in the lower half plane with values in $\mathcal{L}(L^2, \dot{H}_\omega^1)$.

It remains to prove the quantitative bound. Given $f \in L^2$, we split it into two regions,

$$f = f_{in} + f_{out} := \beta_{in} f + \beta_{out} f$$

where

$$\beta_{in} = \beta_{<|\Im\omega|^{-1}}, \quad \beta_{out} = \beta_{>|\Im\omega|^{-1}}.$$

For the inner part we have the estimate

$$\|R_\omega f_{in}\|_{\mathcal{LE}_\omega^1} \lesssim \|f_{in}\|_{\mathcal{LE}^*} \lesssim |\Im\omega|^{-\frac{1}{2}} \|f\|_{L^2},$$

which suffices in the region $\{|x| \lesssim |\Im\omega|^{-1}\}$. To estimate $R_\omega f_{in}$ in the outer region we truncate it there, and set $u = \beta_{out} R_\omega f_{in}$. In the support of u we are in the small asymptotically flat regime, where we know the resolvent bound. Indeed, by (2.5), it suffices to consider $\Im\omega$ in a bounded strip such as $(-2\beta, 0)$, and we then simply choose the constants sufficiently large depending on the constant β in (2.4). Thus, commuting β_{out} with P we have

$$\|u\|_{\dot{H}_\omega^1} \lesssim |\Im\omega|^{-1} \|P_\omega u\|_{L^2} \lesssim |\Im\omega|^{-\frac{1}{2}} \|R_\omega f_{in}\|_{\mathcal{LE}_\omega^1} + |\Im\omega|^{-1} \|f_{in}\|_{L^2} \lesssim |\Im\omega|^{-1} \|f_{in}\|_{L^2}.$$

For the outer part we proceed in two steps. First we introduce an auxiliary operator \tilde{P} which is a small asymptotically flat perturbation of \square which agrees with P in the outer region $\{|x| \gtrsim |\Im\omega|^{-1}\}$. We use its associated resolvent to define the outer approximate solution

$$u_{out} = \beta_{out} \tilde{R}_\omega f_{out}$$

Commuting to estimate the truncation error, we have $P_\omega u_{out} - f_{out}$ supported in the region $\{|x| \lesssim |\Im\omega|^{-1}\}$ and satisfying the bound

$$\|P_\omega u_{out} - f_{out}\|_{L^2} \lesssim |\Im\omega| \|\tilde{R}_\omega f_{out}\|_{\dot{H}_\omega^1} + \|f_{out}\|_{L^2} \lesssim \|f_{out}\|_{L^2}.$$

Hence for this error we can use the same argument as for the inner part.

As a consequence of the uniform energy resolvent bound (2.7), it follows that the solutions to the linear P equation have subexponential energy decay, see Proposition 2.1 (c). This implies that the resolvent formula (2.3) holds pointwise in the lower half-space. Suppose that u solves the forward problem $Pu = f \in L_{x,t}^2$, with f supported in $t \geq 0$. Conjugating by a decaying exponential $e^{-\sigma t}$ we obtain the equation

$$P(x, D_x, D_t - i\sigma)v = g, \quad v = e^{-\sigma t}u, \quad g = e^{-\sigma t}f.$$

Then by the Fourier inversion formula applied on the line $\Im\omega = -\sigma$ we obtain the representation formula for the time Fourier transform

$$\hat{v}(x, \tau) = R_{\tau - i\sigma} \hat{g}(x, \tau).$$

We claim that the resolvent LE bound (8.2) implies the estimate

$$(8.4) \quad \|e^{-\sigma t}u\|_{LE^1} = \|v\|_{LE^1} \lesssim \|g\|_{LE^*} = \|e^{-\sigma t}f\|_{LE^*}.$$

Given the dyadic ℓ^∞ structure of the LE^1 norm, respectively the ℓ^1 structure of the LE^* norm, it suffices to prove the bound in a fixed dyadic region for v , assuming that g is also supported in a fixed (possibly different) dyadic region. But then (8.4) follows from (8.2) directly by Plancherel's theorem.

Letting $\sigma \rightarrow 0$ in (8.4) we obtain the unweighted local energy bound

$$\|u\|_{LE^1} \lesssim \|f\|_{LE^*}$$

for the solution u to the forward problem $Pu = f$ with zero Cauchy data. Finally, using the extension procedure in Section 7.1, one can easily add nontrivial finite energy Cauchy data, and then use energy estimates to control the uniform energy norms and obtain (8.1).

8.2. Real resonances and the limiting absorption principle. Here we consider the question of extending the resolvent to the real axis. Our main result clarifies the definition of the resolvent on the real axis.

Proposition 8.1. *Assume that the LE resolvent bounds (8.2) hold uniformly in the lower half-space H near some real $\omega_0 \neq 0$. Then the limit*

$$R_{\omega_0} f = \lim_{\omega \rightarrow \omega_0} R_\omega f, \quad f \in \mathcal{LE}^*,$$

converges strongly on compact sets. Further, the resolvent bound (8.2) holds at ω_0 and $R_{\omega_0} f$ satisfies the outgoing radiation condition (2.10).

Proof. We first prove that the bound

$$(8.5) \quad \|u\|_{\mathcal{LE}_{\omega_0}^1} \lesssim \|P_{\omega_0}u\|_{\mathcal{LE}^*}$$

holds for all u for which both norms above are finite and which satisfy the outgoing radiation condition (2.10). Suppose $P_{\omega_0}u = f$. Set $u_\epsilon = e^{-\epsilon r}u$. Then we have

$$\begin{aligned} P_{\omega_0 - i\epsilon}u_\epsilon &= e^{-\epsilon r}f + e^{-\epsilon r} \left(-\epsilon^2(g^{rr} + g^{00})u + 2\epsilon(g^{rr}\partial_r u - i\omega_0 g^{00}u) \right. \\ &\quad \left. + \epsilon \mathcal{O}(|g - I|)|\nabla u| + \epsilon \mathcal{O}(|A_\alpha| + |\partial g| + |g - I|)|u| + \frac{2\epsilon}{r}u \right). \end{aligned}$$

The bound (8.5) follows by applying the resolvent bound for $\omega_0 - i\epsilon$ and passing to the limit. For that we need to show that

$$\lim_{\epsilon \rightarrow 0} P_{\omega_0 - i\epsilon}u_\epsilon - e^{-\epsilon r}f = 0 \quad \text{in } \mathcal{LE}^*$$

All terms in the difference decay due to the ℓ^1 dyadic summation, except for the third term where we also use the radiation condition.

Next we show that for each $f \in \mathcal{LE}^*$ there exists $u \in \mathcal{LE}_{\omega_0}^1$, satisfying the outgoing radiation condition, so that $P_{\omega_0}u = f$. Take a sequence $H \ni \omega_n \rightarrow \omega_0$ and corresponding solutions u_n to $P_{\omega_n}u_n = f$. Then we obtain u as a weak limit of a subsequence of u_n , and the uniform \mathcal{LE}^1 bound for u_n transfers to u . Further, by elliptic regularity the sequence u_n is locally in H^2 , therefore strong H^1 convergence follows on compact sets. It remains to show that the outgoing radiation condition (2.10) holds for the limit.

Here we shall again use a positive commutator argument. A related, though easier, calculation for the Schrödinger equation can be found in [16]. For convenience, we abbreviate $\Delta_g = \Delta_{g,0}$, where the latter notation is from (2.1). And we shall again assume without loss that $g^{00} = -1$. We record that

$$\begin{aligned} (8.6) \quad &-2\Im\langle Qu, (\Delta_g - (\omega_0 - i\epsilon)^2 + (\omega_0 - i\epsilon)(g^{0k}D_k + D_k g^{0k}))u \rangle \\ &= \langle i[\Delta_g + \omega_0(g^{0k}D_k + D_k g^{0k}), Q]u, u \rangle + 4\epsilon\omega_0\langle Qu, u \rangle \\ &\quad - \epsilon\langle Qu, (g^{0k}D_k + D_k g^{0k})u \rangle - \epsilon\langle (g^{0k}D_k + D_k g^{0k})u, Qu \rangle \end{aligned}$$

for a symmetric operator Q . We set

$$Q = f(R) \left(\frac{x_i g^{ij}}{R} D_j + g^{0k} D_k + \omega_0 \right) + \left(D_i \frac{g^{ij} x_j}{R} + D_k g^{0k} + \omega_0 \right) f(R), \quad R^2 = x_i g^{ij} x_j.$$

The choice of multiplier is motivated by the last three terms in (8.6). Indeed, we have

$$\begin{aligned} \omega_0 Q &= \left(D_k \frac{g^{kl} x_l}{R} + \omega_0 \right) f(R) \left(\frac{x_i g^{ij}}{R} D_j + \omega_0 \right) + \left(D_i - D_l \frac{g^{lk} x_k x_i}{R^2} \right) g^{ij} f(R) \left(D_j - \frac{x_j x_m g^{mn}}{R^2} D_n \right) \\ &\quad - \frac{f(R)}{2} (\Delta_g - \omega_0^2 + \epsilon^2 + (\omega_0 - i\epsilon)(g^{0k}D_k + D_k g^{0k})) - (\Delta_g - \omega_0^2 + \epsilon^2 + (\omega_0 - i\epsilon)(g^{0k}D_k + D_k g^{0k})) \frac{f(R)}{2} \\ &\quad - i\epsilon \frac{f(R)}{2} (g^{0k}D_k + D_k g^{0k}) - i\epsilon (g^{0k}D_k + D_k g^{0k}) \frac{f(R)}{2} + f(R)\epsilon^2 + \frac{1}{2}(\Delta_g f(R)). \end{aligned}$$

Using this, we observe

$$\begin{aligned} &4\epsilon\omega_0\langle Qu, u \rangle - \epsilon\langle Qu, (g^{0k}D_k + D_k g^{0k})u \rangle - \epsilon\langle (g^{0k}D_k + D_k g^{0k})u, Qu \rangle \\ &= 4\epsilon\langle \left(D_k \frac{g^{kl} x_l}{R} - D_k g^{0k} + \omega_0 \right) f(R) \left(\frac{x_i g^{ij}}{R} D_j - g^{0j} D_j + \omega_0 \right) u, u \rangle + 4\epsilon\langle D_k g^{0k} f(R) g^{0j} D_j u, u \rangle \\ &\quad + 4\epsilon\langle \left(D_i - D_l \frac{g^{lk} x_k x_i}{R^2} \right) g^{ij} f(R) \left(D_j - \frac{x_j x_m g^{mn}}{R^2} D_n \right) u, u \rangle + 4\epsilon^3\langle f(R)u, u \rangle \\ &\quad - 4\epsilon\Re\langle (\Delta_g - \omega_0^2 + \epsilon^2 + (\omega_0 - i\epsilon)(g^{0k}D_k + D_k g^{0k}))u, f(R)u \rangle + \langle E_1 u, u \rangle + \langle E_2 u, u \rangle. \end{aligned}$$

Here, upon fixing $f(R) = \frac{R}{R+2^j}$ and noting that $f(R)|_{R \approx 2^k} \approx 2^{k-j}$ when $k < j$, we have that the first four terms on the right are non-negative and that the errors satisfy

$$\begin{aligned} \langle E_1 u, u \rangle|_{|x| \approx 2^k} &= \epsilon^2 2^{-(k-j)^-} \mathcal{O}(c_k + 2^{-k}) \|(u, \nabla u)\|_{L^2}^2, \\ \langle E_2 u, u \rangle|_{|x| \approx 2^k} &= 2^{-(k-j)^-} \mathcal{O}(c_k + 2^{-k}) (\|u\|_{\mathcal{LE}}^2 + \|\nabla u\|_{\mathcal{LE}}^2). \end{aligned}$$

On the other hand, for the purposes of the commutator, up to error terms that are easily handled, we may replace g^{ij} by the identity matrix which yields

$$\begin{aligned} i[\Delta_g + \omega_0(g^{0k}D_k + D_k g^{0k}), Q] &= 2D\left(2\frac{f(R)}{R} - f'(R)\right)D - 2Dx\left(2\frac{f(R)}{R^3} - \frac{f'(R)}{R^2}\right)x \\ &\quad + f'(R)(\Delta_g - \omega_0^2 + \varepsilon^2) + (\Delta_g - \omega_0^2 + \varepsilon^2)f'(R) + 2(Dx + \omega_0 r)\frac{f'(R)}{rR}(xD + r\omega_0) - 2f'(R)\varepsilon^2 + E_2. \end{aligned}$$

This choice of f gives positive coefficients in the first two factors above. Moreover, we have that $f'(R)|_{R \approx 2^j} \approx 2^{-j}$.

We may thus combine the two calculations above to see that

$$\begin{aligned} 2^{-j}\|(\partial_r + i\omega_0)u\|_{L^2(|x| \approx 2^j)}^2 &\lesssim \Re\langle (\Delta_g - (\omega_0 - i\varepsilon)^2 + (\omega_0 - i\varepsilon)(g^{0k}D_k + D_k g^{0k}))u, (iQ - 4\varepsilon f(R) + 2f'(R))u \rangle \\ &\quad + \langle E_1 u, u \rangle + \langle E_2 u, u \rangle \\ &\lesssim \sum_{k \geq 0} 2^{-(k-j)^-} \left(2^{j/2}\|P_{\omega_0 - i\varepsilon}u\|_{L^2(|x| \approx 2^j)}\right) \left(2^{-j/2}\|(u, \nabla u)\|_{L^2(|x| \approx 2^j)}\right) \\ &\quad + (\|(u, \nabla u)\|_{\mathcal{LE}}^2 + \varepsilon^2\|u\|_{L^2}^2) \left(\sum_{k \geq 0} 2^{-(k-j)^-} (c_k + 2^{-k})\right). \end{aligned}$$

Applying this estimate to the sequence $u_n = R_{\omega_n}f$ and utilizing the known resolvent energy and resolvent LE bounds shows that the outgoing radiation condition holds.

Finally, we observe that by (8.5) the limit u of the subsequence of u_n is unique; therefore the full sequence u_n converges to u strongly in H_{loc}^1 . \square

8.3. Nonzero resonances and LE resovent bounds. Here we prove the equivalence of the three statements in Proposition 2.5. We do this in several steps:

1. A simple case. Here we consider a simple situation, namely when $A = 0$, $V = 0$. Then P is self-adjoint, and the conserved energy associated to P is positive definite. Then by Theorem 2.15, local energy decay holds for P . Thus, the resolvent extends to the real axis with uniform bounds.

2. A perturbative formulation. Given a stationary operator P , we denote by \tilde{P} its principal part, with $A = 0$ and $V = 0$. Then the resolvent \tilde{R}_ω of \tilde{P}_ω is defined globally in the lower half-space, as well as on the real line. We seek a solution u to $P_\omega u = f$ of the form $u = \tilde{R}_\omega g$. Denoting $Q_\omega = P_\omega - \tilde{P}_\omega$, we rewrite our equation as

$$(I + Q_\omega \tilde{R}_\omega)g = f.$$

The operator $Q_\omega \tilde{R}_\omega$ is compact, and depends analytically on ω in the lower half-space and continuously on the real line.

If $-\Im \omega$ is large enough then $I + Q_\omega \tilde{R}_\omega$ is invertible. Hence it has zero Fredholm index everywhere. Thus statements (a), (b), (c) in Proposition 2.5 are equivalent by standard Fredholm theory.

8.4. Stability in LE resolvent bounds. Here we prove the stability statement in Proposition 2.5. Precisely, we assume that the resolvent bounds (8.2) for P hold at some real ω_0 , and show that they must hold uniformly in a neighbourhood of ω_0 , not only for P but also for small AF perturbations of P .

The proof is by contradiction. Suppose we have a sequence of operators P_n converging to P in the AF norm, a sequence of numbers $\omega_n \in H$ converging to ω , and a sequence of functions u_n so that

$$\|u_n\|_{\mathcal{LE}_{\omega_n}^1} = 1, \quad \|P_n u_n\|_{\mathcal{LE}^*} \rightarrow 0.$$

By elliptic estimates the sequence u_n is also locally bounded in H^2 ; therefore on a subsequence it converges to some $u \in \mathcal{LE}_{\omega_0}^1$ so that $P_{\omega_0} u = 0$. To reach a contradiction we need the following two observations:

(a) By exterior local energy resolvent bounds, i.e. those obtained by mimicking the procedure that yielded (8.3) when one starts instead from (3.1), a nontrivial portion of u_n is localized to a compact set, $\|u_n\|_{H_{<2R_0}^1} \gtrsim 1$. This shows that $u \neq 0$.

(b) By the same argument as in the previous section, the function u satisfies the outgoing radiation condition (2.10).

We remark that this argument implies in effect the slightly stronger conclusion

$$\limsup_{n \rightarrow \infty} \|R_n(\omega_n)\|_{\mathcal{LE}^* \rightarrow \mathcal{LE}_{\omega_n}^1} \lesssim \|R(\omega_0)\|_{\mathcal{LE}^* \rightarrow \mathcal{LE}_{\omega_0}^1}.$$

8.5. High frequency analysis. Here we prove that the high frequency bound in Theorem 2.11 implies the uniform LE resolvent bound for large ω . The case $-\Im\omega \gg 1$ follows directly from energy estimates; see (2.5). Hence we assume that $0 < -\Im\omega \lesssim 1$.

Suppose that $P_\omega u = f$. Then $v = e^{i\omega t}u$ solves $Pv = g$ with $g = e^{i\omega t}f$. Applying the bound (2.14) to v in $(-\infty, 0]$ we obtain

$$\|u\|_{\mathcal{LE}^1} + |\omega| \|u\|_{\mathcal{LE}} + |\Im\omega|^{\frac{1}{2}} \|u\|_{\dot{H}_\omega^1} \lesssim \|\langle x \rangle^{-2} u\|_{\mathcal{LE}} + \|f\|_{\mathcal{LE}^* + |\Im\omega|^{\frac{1}{2}} L^2}.$$

If ω is large enough then the first term on the right is absorbed on the left, and the conclusion follows.

8.6. The general case. Here we prove Theorem 2.12 (a); part (b) will then follow due to Theorem 2.6. Since there are no eigenvalues in the lower half-space, the resolvent is holomorphic in H . Combining this with the above high frequency bound, it follows that we must have an estimate of the form

$$\|R_\omega\|_{L^2 \rightarrow \dot{H}_\omega^1} \leq c(\Im\omega), \quad \Im\omega < 0.$$

This in turn implies the LE resolvent bound outside any neighbourhood of the real line. Now we consider a strip near the real line. We have three cases:

(i) $|\omega|$ is sufficiently large. Then, as discussed in the previous subsection, the LE resolvent bound (8.2) holds.

(ii) $|\omega|$ is sufficiently small. Then from Theorem 6.1, arguing as in the previous subsection, we obtain again (8.2).

(iii) $|\omega|$ is away from zero or infinity. Here from hypothesis in case (a) and from Theorem 2.6, we know that there are no real resonances. Then Proposition 2.5 shows that (8.2) holds for ω sufficiently close to the real axis.

The proof of (8.2), and thus of Theorem 2.12, is concluded.

9. SPECTRAL THEORY AND EXPONENTIAL TRICHOTOMIES

This section is concerned with deriving Theorems 2.13 and 2.17 from Theorems 2.15 and 2.16. First we give a different perspective for the spectral theory for symmetric $P(x, D)$ in the time independent case. This material is more or less standard (e.g. [2], [14], [15]).

Next, we use the time independent analysis to derive a perturbative result which holds on large time intervals assuming the symbol of $P(t, x, D)$ has sufficiently slow time variation. Based on this we string together the local analysis via a Perron type argument to globally construct the stable/unstable/center subspaces $S^{\pm, 0}(t)$. This last step follows closely work of Coppel [5].

9.1. Spectral theory in the time independent case. Here we assume that the operator P is symmetric, and we write it in the form

$$P = P_0 + B(x, D_x)D_t + g^{00}D_t^2$$

where

$$\begin{aligned} P_0 &= (D + A)_j g^{jk} (D + A)_k + W(x, D_x), \\ W(x, D_x) &= A_0 g^{0k} (D_k + A_k) + (D_k + A_k) A_0 g^{0k} + g^{00} A_0^2 + V(x), \\ B(x, D_x) &= g^{0k} (D_k + A_k) + (D_k + A_k) g^{0k} + 2A_0 g^{00}. \end{aligned}$$

Using this expression we recast the equation $Pu = f$ as a system, setting

$$U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} U_0 \\ U_1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ i(g^{00})^{-1}f \end{pmatrix}.$$

Then we write the inhomogeneous wave equation $Pu = f$ as a Schrödinger type equation:

$$(9.1) \quad D_t U = \mathcal{H}U + F, \quad \text{where } \mathcal{H} = \frac{1}{i} \begin{pmatrix} 0 & 1 \\ (g^{00})^{-1}P_0 & -i(g^{00})^{-1}B \end{pmatrix}.$$

The homogeneous evolution preserves the energy form on $\dot{H}^1 \times L^2$ defined by:

$$(9.2) \quad E[U] = \langle U, U \rangle_E, \quad \text{where} \quad \langle U, \Psi \rangle_E = \int_{\mathbb{R}^3} P_0 U_0 \overline{\Psi_0} - g^{00} U_1 \overline{\Psi_1} dx.$$

One readily checks that \mathcal{H} is a symmetric operator with respect to this energy form, i.e.

$$\langle \mathcal{H}U, \Psi \rangle_E = \langle U, \mathcal{H}\Psi \rangle_E$$

on an appropriate dense domain for \mathcal{H} (say $H^2 \times H^1$), and that \mathcal{H} extends to a closed operator on $\dot{H}^1 \times L^2$.

To relate the spectral theory of \mathcal{H} with that for P , we remark that the solutions to the linear eigenvalue problem $\mathcal{H}U_\omega = \omega U_\omega$ are of the form $U_\omega = (u_\omega, i\omega u_\omega)$, where u_ω solves $P_\omega u_\omega = 0$. More generally, we can relate the resolvent of \mathcal{H} , which we denote by \mathcal{R}_ω , with the resolvent R_ω of P as follows:

$$(9.3) \quad \mathcal{R}_\omega = \begin{pmatrix} -R_\omega(B + \omega g^{00}) & iR_\omega g^{00} \\ i(-\omega R_\omega(B + \omega g^{00}) + I) & -\omega R_\omega g^{00} \end{pmatrix}.$$

As a straightforward consequence of Proposition 2.1 we have the following

Proposition 9.1. *Assume that P is asymptotically flat. Then the resolvent $\mathcal{R}_\omega \in \mathcal{L}(\dot{H}^1 \times L^2)$ is meromorphic in the lower half-space, with the same poles as R_ω . Further, for each ω in the lower half-space the bounds (2.7) and*

$$(9.4) \quad \|\mathcal{R}_\omega\|_{\mathcal{L}(\dot{H}^1 \times L^2)} \lesssim |\Im \omega|^{-1}$$

are equivalent.

The non-trivial spectral theory of \mathcal{H} in general stems from the fact that $\langle \cdot, \cdot \rangle_E$ may not be positive definite. In fact, if this form has a negative index then there exist non-trivial null vectors $\langle U, U \rangle_E = 0$, and one easily checks that solutions to $\mathcal{H}U_\omega = \omega U_\omega$ with non-real ω must have this property. On any plane where $\langle \cdot, \cdot \rangle_E$ is trivial (an “isotropic subspace”) the self-adjointness of \mathcal{H} gives no additional information so one can expect the eigenspace structure of \mathcal{H} to be more or less arbitrary there, unless of course more structure on P is assumed (e.g. if $A_0 = g^{0k} = 0$ then non-real eigenvalues must be purely imaginary).

The interesting fact is that in the case we are considering, i.e. where P is symmetric and the zero frequency estimate (2.13) holds, the discrete spectral theory of \mathcal{H} is completely described by two maximal \mathcal{H} invariant isotropic subspaces S^\pm whose dimension must be equal to the negative index of $\langle \cdot, \cdot \rangle_E$. Then one can decompose $\dot{H}^1 \times L^2 = (S^- + S^+) \oplus_E S^0$, where \oplus_E denotes an orthogonal direct sum with respect to $\langle \cdot, \cdot \rangle_E$. The crucial fact now is that S^0 is a closed sub-space of $\dot{H}^1 \times L^2$ on which $\langle \cdot, \cdot \rangle_E$ becomes positive definite. In particular $\langle \cdot, \cdot \rangle_E$ restricted to S^0 defines a *positive conserved* energy norm which is equivalent to $\|\cdot\|_{\dot{H}^1 \times L^2}$. We sum this information up as follows:

Proposition 9.2 (Spectral theory for symmetric $P(x, D)$). *Let $P(x, D)$ be a symmetric, stationary, asymptotically flat wave operator. Let \mathcal{H} be the Hamiltonian matrix of P as defined on line (9.1). Suppose in addition that the zero spectral estimate (2.13) holds for some $K_0 > 0$. Then one has the following:*

- i) *(The pair $(\dot{H}^1 \times L^2, \langle \cdot, \cdot \rangle_E)$ is a Pontryagin space) The energy form $\langle \cdot, \cdot \rangle_E$ is nondegenerate on the energy space $\mathcal{E} = \dot{H}^1 \times L^2$, and we can write it as orthogonal direct sum $\mathcal{E} = \mathcal{E}^- \oplus_E \mathcal{E}^+$ where \mathcal{E}^- has dimension $\kappa \in \mathbb{N}$, and where $\|U\|_E^2 = \langle U^+, U^+ \rangle_E - \langle U^-, U^- \rangle_E \approx \|U\|_{\dot{H}^1 \times L^2}^2$. Here $U = U^- + U^+$ is the decomposition of U along \mathcal{E}^\pm .*
- ii) *(Pontryagin’s Theorem) There exists two (maximal) κ -dimensional isotropic \mathcal{H} invariant subspaces $S^\pm \subseteq \mathcal{E}$, and an $\alpha > 0$ such that $\Im(\omega) \geq \alpha$ for all $\omega \in \text{spec}(\mathcal{H}|_{S^-})$ and $\Im(\omega) < -\alpha$ for all $\omega \in \text{spec}(\mathcal{H}|_{S^+})$. Moreover $\mathcal{E}_{disc} = S^- + S^+$ accounts for the entire discrete spectrum of \mathcal{H} , and $\langle \cdot, \cdot \rangle_E$ restricted to \mathcal{E}_{disc} is a nondegenerate form of index $(-\kappa, \kappa)$.*
- iii) *(Coercivity away from S^\pm) Finally, if we write $\mathcal{E}_{ac} = \mathcal{E}_{disc}^\perp$ then $\langle \cdot, \cdot \rangle_E$ restricted to \mathcal{E}_{ac} gives rise to a norm equivalent to $\dot{H}^1 \times L^2$:*

$$(9.5) \quad \langle U, U \rangle_E \geq c \|U\|_{\dot{H}^1 \times L^2}^2, \quad U \in \mathcal{E}_{ac}.$$

Proof of Proposition 9.2. This material is pretty standard and essentially covered in [2], [14], [15], etc. For Part i) considerations are similar to those in [15]. For Part ii), which is the main invariant subspace theorem of Krein space theory, see in particular Theorem 3.2 of [2]. Part iii) uses some basic geometry of Krein spaces (see Section 1 of [14]), and the closed graph theorem. \square

To the subspaces S^\pm we can associate commuting projectors P^\pm in a standard fashion, using the resolvent:

$$(9.6) \quad P^\pm = \int_{\gamma^\pm} \mathcal{R}_\omega d\omega$$

where γ^\pm are contours enclosing the spectrum of \mathcal{H} in the lower, respectively the upper half-space. Similarly, the wave flow on S^\pm is expressed as

$$(9.7) \quad e^{it\mathcal{H}} P^\pm = \int_{\gamma^\pm} e^{it\omega} \mathcal{R}_\omega d\omega.$$

Now we are able to complete the proof of Theorem 2.13. The key element in the proof is the following.

Proposition 9.3. *Let P be stationary, symmetric, asymptotically flat, nontrapping and satisfying the zero resolvent bound (2.13). Then*

a) We have the following estimates

$$(9.8) \quad \|P^\pm\|_{\mathcal{L}(\dot{H}^1 \times L^2)} \lesssim 1$$

$$(9.9) \quad \begin{aligned} \|e^{it\mathcal{H}}|_{S^+}\|_{\mathcal{L}(\dot{H}^1 \times L^2)} &\lesssim e^{\alpha t}, \quad t < 0, & \|e^{it\mathcal{H}}|_{S^+}\|_{\mathcal{L}(\dot{H}^1 \times L^2)} &\lesssim e^{\beta t}, \quad t \geq 0 \\ \|e^{it\mathcal{H}}|_{S^-}\|_{\mathcal{L}(\dot{H}^1 \times L^2)} &\lesssim e^{-\beta t}, \quad t < 0, & \|e^{it\mathcal{H}}|_{S^-}\|_{\mathcal{L}(\dot{H}^1 \times L^2)} &\lesssim e^{-\alpha t}, \quad t \geq 0 \end{aligned}$$

with implicit constants as well as α and β depending only on our parameters M_0, R_0, T_0 and K_0 .

b) The constant c in (9.5) also depends only on our parameters M_0, R_0, T_0 and K_0 .

c) The operators P^\pm and $e^{it\mathcal{H}} P^\pm$ have a Lipschitz dependence on the coefficients of P in the asymptotically flat topology.

We remark that Theorem 2.13 follows immediately. The exponential bounds for S^+ and S^- follow immediately from the above bounds. On the other hand, solutions in S_0 have bounded energy by part (iii) of Proposition 9.2. Hence, by Theorem 2.16 we have both uniform energy bounds and local energy decay. We further remark that on S_0 the energy and the form $\langle \cdot, \cdot \rangle_E$ are equivalent.

Proof of Proposition 9.3. a) By symmetry it suffices to consider the “+” case. The result in part (a) will follow from the integral representations (9.6), (9.7) if we can find a contour rectangle $R = [-M, M] \times [-\alpha, -\beta]$ which contains all the spectrum of P in the lower half-space, so that the estimate (2.7) holds uniformly on the boundary γ^+ of R .

The parameter β is easily found as a consequence of the energy estimates, see (2.5). The parameter M is found as a consequence of the high frequency bounds in Section 8.5. Finally, to find α we need the full strength of the two point LE bound in Theorem 2.15. Indeed, suppose that $P_\omega u = f$. Then $v = e^{i\omega t} u$ solves $Pv = g$ with $g = e^{i\omega t} f$. Hence applying the two point LE estimate to v on $(-\infty, 0]$ we obtain

$$\|u\|_{\mathcal{LE}^1} + |\omega| \|u\|_{\mathcal{LE}} \lesssim \|f\|_{\mathcal{LE}^*} + |\Im \omega|^{\frac{1}{2}} (\|u\|_{\dot{H}^1} + |\omega| \|u\|_{L^2}).$$

On the other hand, for $R > R_0$ we have the energy relation (1.10), which combined with (1.11) gives

$$|\Im \omega| (\|u\|_{\dot{H}^1}^2 + |\omega|^2 \|u\|_{L^2}^2) \lesssim |\Im \omega| \|u\|_{L_{comp}^2}^2 + |\omega| \|f\|_{\mathcal{LE}^* + |\Im \omega|^{\frac{1}{2}} L^2} \|u\|_{\mathcal{LE} \cap |\Im \omega|^{-\frac{1}{2}} L^2}.$$

Combining the two bounds we obtain

$$\|u\|_{\mathcal{LE}^1} + |\omega| \|u\|_{\mathcal{LE}} + |\Im \omega|^{\frac{1}{2}} (\|u\|_{\dot{H}^1} + |\omega| \|u\|_{L^2}) \lesssim \|f\|_{\mathcal{LE}^* + |\Im \omega|^{\frac{1}{2}} L^2}$$

provided that $|\Im \omega|$ is sufficiently small. Here we have used a Hardy inequality to control $\|u\|_{L_{comp}^2}$ by $\|u\|_{\mathcal{LE}^1}$. This yields the desired resolvent energy bound.

b) The form $\langle \cdot, \cdot \rangle_E$ is semipositive on $(S^+)^\perp$, with degeneracy only on $S^+ \subset (S^+)^\perp$, and similarly on $(S^-)^\perp$. Thus the constant c can degenerate only if the subspaces S^+ and S^- are very close. However, this is prohibited by the bound in part (a) on the projectors P^+ and P^- .

c) This is a straightforward consequence of the similar property of the resolvent bound along the curve γ^+ , which in turn is a consequence of the enhanced resolvent bound

$$(9.10) \quad |\omega|^{-1} \|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^2} + |\omega| \|u\|_{L^2} \lesssim |\Im \omega|^{-1} \|P_\omega u\|_{L^2}$$

which is an elliptic consequence of the standard one. \square

9.2. Perturbation theory of S^\pm . As a preliminary step toward the proof of Theorem 2.17, in this subsection we prove a local version of Theorem 2.17 for ϵ almost stationary, almost symmetric operators $P(t, x, D)$.

The spectral decomposition in the previous section is associated to symmetric operators. In order to use it here, we denote by $P_{sym}(t, x, D)$ the symmetrized operators $P(t, x, D)$ (i.e. where A and V are replaced by their real parts). Correspondingly, we define the time dependent subspaces $S^{\pm,0}(t)$ and projectors $P^{\pm,0}(t)$.

Proposition 9.4 (Local stable/unstable/center manifold construction). *Let P satisfy all of the assumptions of Theorem 2.16. Then there exists a time threshold $T_* = T_*(\epsilon)$, such that $T_* \rightarrow \infty$ as $\epsilon \rightarrow 0$, so that the estimates (2.19)–(2.21) hold for the evolution $\mathcal{F}(t, t_0)$ of $P(t, x, D)$ for all $t, s \in J_{t_0} = [t_0, t_0 + T_*]$. Moreover, one has:*

$$(9.11) \quad \|P^{\pm,0}(t)\mathcal{F}(t, t_0) - \mathcal{F}(t, t_0)P^{\pm,0}(t_0)\|_{\dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2} \lesssim \epsilon e^{C|t-t_0|}, \quad t \in J_{t_0}.$$

We remark that the time threshold T^* is chosen so that $\epsilon = e^{-CT^*}$ for a large constant C .

Proof. We begin with a simple observation, namely that

$$\|\mathcal{H} - \mathcal{H}_{sym}\|_{\dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2} = o(\epsilon).$$

Thus, by Gronwall, one can harmlessly replace P by P_{sym} and \mathcal{H} by \mathcal{H}_{sym} in the proposition.

Secondly, consider initial data $U(t_0) \in S^{\pm,0}(t_0)$. Then we claim that $e^{i(t-t_0)\mathcal{H}(t_0)}$ is a good ϵ -approximate solution. This is easy to see, since

$$\|\mathcal{H}(t) - \mathcal{H}(t_0)\|_{\dot{H}^2 \cap \dot{H}^1 \times \dot{H}^1 \cap L^2 \rightarrow \dot{H}^1 \times L^2} \lesssim \epsilon(1 + |t - t_0|),$$

while

$$\|P^{\pm,0}\|_{\dot{H}^1 \times L^2 \rightarrow \dot{H}^2 \cap \dot{H}^1 \times \dot{H}^1 \cap L^2} \lesssim 1.$$

Thus we obtain

$$\|(D_t - \mathcal{H})e^{i(t-t_0)\mathcal{H}(t_0)}U(t_0)\|_{\dot{H}^1 \times L^2} \lesssim \epsilon e^{C|t-t_0|} \|e^{i\mathcal{H}(t-t_0)}U(t_0)\|_{\dot{H}^1 \times L^2},$$

which by the Duhamel formula and an estimate akin to (2.4) shows that

$$(9.12) \quad \|(\mathcal{F}(t, t_0) - e^{i(t-t_0)\mathcal{H}(t_0)})U(t_0)\|_{\dot{H}^1 \times L^2} \lesssim \epsilon e^{C|t-t_0|} \|e^{i\mathcal{H}(t-t_0)}U(t_0)\|_{\dot{H}^1 \times L^2}.$$

This already proves the bounds (2.19) and (2.20) for $s = t_0$.

We now establish (9.11). For this we write

$$\begin{aligned} & P^{\pm,0}(t)\mathcal{F}(t, t_0)P^{\pm,0}(t_0) - \mathcal{F}(t, t_0)P^{\pm,0}(t_0) \\ &= (P^{\pm,0}(t) - P^{\pm,0}(t_0))\mathcal{F}(t, t_0)P^{\pm,0}(t_0) + (P^{\pm,0}(t_0) - I)(\mathcal{F}(t, t_0) - e^{it\mathcal{H}(t_0)})P^{\pm,0}(t_0). \end{aligned}$$

Using part (c) of Proposition 9.3 for the first term on the right and (9.12) for the second we obtain

$$\|P^{\pm,0}(t)\mathcal{F}(t, t_0)P^{\pm,0}(t_0) - \mathcal{F}(t, t_0)P^{\pm,0}(t_0)\|_{\dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2} \lesssim \epsilon e^{C|t-t_0|}.$$

Then (9.11) follows by time reversal. The bounds (2.19) and (2.20) for $s \neq t_0$ are a direct consequence of (9.11) and the $s = t_0$ case.

It remains to prove (2.21). Due to (9.11) it again suffices to consider the case $s = t_0$. For this we consider the energy conservation relation, which in the time dependent case reads

$$\frac{d}{dt}E(U(t)) \lesssim \epsilon \|U(t)\|_{\dot{H}^1 \times L^2}^2.$$

This implies that

$$|E(U(t)) - E(U(t_0))| \lesssim \epsilon e^{C|t-t_0|} \|U(t_0)\|_{\dot{H}^1 \times L^2}^2.$$

We apply this to $U(t) = \mathcal{F}(t, t_0)P^0(t_0)U_0$, which by (9.11) is well approximated by $P^0(t)\mathcal{F}(t, t_0)U_0$. By part (b) of Proposition 9.3, the (time dependent) energy functional $E(t)$ is uniformly positive definite on the

$S^0(t)$, therefore we can write

$$\begin{aligned}
\|U(t)\|_{\dot{H}^1 \times L^2}^2 &= \|P^0(t)\mathcal{F}(t, t_0)U_0\|_{\dot{H}^1 \times L^2}^2 + O(\epsilon e^{C(t-t_0)})\|U_0\|_{\dot{H}^1 \times L^2}^2 \\
&\approx E(P^0(t)\mathcal{F}(t, t_0)U_0) + O(\epsilon e^{C(t-t_0)})\|U_0\|_{\dot{H}^1 \times L^2}^2 \\
&= E(U(t)) + O(\epsilon e^{C(t-t_0)})\|U_0\|_{\dot{H}^1 \times L^2}^2 \\
&= E(U(t_0)) + O(\epsilon e^{C(t-t_0)})\|U_0\|_{\dot{H}^1 \times L^2}^2 \\
&\approx \|P^0(t_0)U_0\|_{\dot{H}^1 \times L^2}^2 + O(\epsilon e^{C(t-t_0)})\|U_0\|_{\dot{H}^1 \times L^2}^2.
\end{aligned}$$

Hence (2.21) follows, and the proof of the proposition is complete. \square

9.3. Global construction of $P^{\pm,0}$ via Perron sums. We now prove the full version of Theorem 2.17. In light of the energy estimates in Proposition 9.4 and Theorem 2.16 it suffices to prove the following slightly weaker discrete version:

Theorem 9.5 (Discretized version of Theorem 2.17). *Let P be as in the statement of Theorem 2.17. Then there exists a time increment $T_* > 0$ such that $T_* \rightarrow \infty$ as $\epsilon \rightarrow 0$ with the following property. There exists $\kappa \in \mathbb{N}$ and real $C, \alpha, \alpha_0 > 0$, where $\alpha_0 = \alpha_1(\epsilon)$, and two κ dimensional linear subspaces $S^\pm \subseteq \dot{H}^1 \times L^2$ with corresponding finite rank commuting projections P^\pm and $P^0 = I - P^+ - P^-$, such that if $t_n = nT_*$ one has:*

$$(9.13) \quad \|\mathcal{F}(t_n)P^-\mathcal{F}^{-1}(t_m)\|_{\dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2} \leq Ce^{-\alpha(t_n-t_m)}, \quad t_n \geq t_m \geq 0,$$

$$(9.14) \quad \|\mathcal{F}(t_m)P^+\mathcal{F}^{-1}(t_n)\|_{\dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2} \leq Ce^{-\alpha(t_n-t_m)}, \quad t_n \geq t_m \geq 0,$$

$$(9.15) \quad \|\mathcal{F}(t_n)(1 - P^+ - P^-)\mathcal{F}^{-1}(t_m)\|_{\dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2} \leq Ce^{\alpha_0|t_n-t_m|}, \quad \text{all } t_n, t_m \geq 0,$$

where $0 \leq \alpha_0 \ll 1$ is sufficiently small so that estimate (2.17) holds in case (9.15) above. Further, the push-forward of the projectors $P^{\pm,0}$ are close to the time dependent projectors $P^{\pm,0}(t)$, namely

$$(9.16) \quad \|P^{\pm,0}(t_n) - \mathcal{F}(t_n)P^{\pm,0}\mathcal{F}^{-1}(t_n)\|_{\dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2} \lesssim \epsilon.$$

Here the time T^* is as in Proposition 9.4, which also takes care of the estimates within each time interval $[t_n, t_{n+1}]$.

Remark 9.6. The above decomposition of $\dot{H}^1 \times L^2 = S^+ + S^- + S^0$ is not uniquely defined. Precisely, it is the subspaces S^- and $S^- + S^0$ which are uniquely determined, but S^+ and S^0 are not. In our construction below we select these subspaces uniquely by adding the supplementary requirement that $S^+ = S^+(0)$ and $S^+ + S^0 = S^+(0) + S^0(0)$. We further remark that this ambiguity is due to the fact that our set-up is on the positive time line. One also has a similar result that corresponds to the full time line, and there all subspaces S^+ , S^- and S^0 are uniquely determined.

The proof of the above Theorem will follow from Proposition 9.4 and the following abstract Perron type construction for discrete time flows on Banach spaces:

Proposition 9.7. *Let X be a Banach space, and let $\mathcal{G}_n \in \mathcal{B}(X)$ be a sequence of invertible operators, with $\mathcal{G}_0 = I$, which induces an exponential trichotomy:*

$$(9.17) \quad \|\mathcal{G}_n Q^- \mathcal{G}_m^{-1}\|_{\mathcal{B}(X)} \leq Ce^{-\gamma(n-m)}, \quad n \geq m \geq 0,$$

$$(9.18) \quad \|\mathcal{G}_m Q^+ \mathcal{G}_n^{-1}\|_{\mathcal{B}(X)} \leq Ce^{-\gamma(n-m)}, \quad n \geq m \geq 0,$$

$$(9.19) \quad \|\mathcal{G}_n(1 - Q^- - Q^+)\mathcal{G}_m^{-1}\|_{\mathcal{B}(X)} \leq Ce^{\gamma_0|n-m|}, \quad n, m \geq 0,$$

for a pair of projection matrices Q^\pm with $Q^+Q^- = Q^-Q^+ = 0$, where we also assume $1 \leq \gamma_0 \ll \gamma$. Let \mathcal{F}_n be another sequence of bounded operators, with $\mathcal{F}_0 = I$, and such that:

$$(9.20) \quad \|\mathcal{F}_n \mathcal{F}_{n-1}^{-1} - \mathcal{G}_n \mathcal{G}_{n-1}^{-1}\|_{\mathcal{B}(X)} \leq \epsilon,$$

for $\epsilon > 0$ sufficiently small. Then there exist commuting projections P^\pm with $\|P^\pm - Q^\pm\|_{\mathcal{B}(X)} \lesssim \epsilon$ such that:

$$(9.21) \quad \|\mathcal{F}_n P^- \mathcal{F}_m^{-1}\|_{\mathcal{B}(X)} \leq C' e^{-\gamma'(n-m)}, \quad n \geq m \geq 0,$$

$$(9.22) \quad \|\mathcal{F}_m P^+ \mathcal{F}_n^{-1}\|_{\mathcal{B}(X)} \leq C' e^{-\gamma'(n-m)}, \quad n \geq m \geq 0,$$

$$(9.23) \quad \|\mathcal{F}_n(1 - P^- - P^+) \mathcal{F}_m^{-1}\|_{\mathcal{B}(X)} \leq C' e^{\gamma'_0 |n-m|}, \quad n, m \geq 0,$$

where $0 \leq \gamma'_0, \gamma'$ and $|\gamma_0 - \gamma'_0| + |\gamma - \gamma'| \lesssim \epsilon$.

To apply Proposition 9.7 to the proof of Theorem 9.5 we let $\mathcal{F}_n = \mathcal{F}(t_n)$ denote the time t_n flow of $P(t, x, D)$ and let $X = \dot{H}^1 \times L^2$. For each time step we “correct” \mathcal{F}_n so it maps $S^\pm(t_n)$ perfectly into $S^\pm(t_{n+1})$. That is we define \mathcal{G}_n inductively via:

$$\mathcal{G}_{n+1} \mathcal{G}_n^{-1} = \sum_{a=\pm, 0} P^a(t_{n+1}) \mathcal{F}_{n+1} \mathcal{F}_n^{-1} P^a(t_n),$$

and set $Q^\pm = P^\pm(0)$. Using estimate (9.11) we have (9.20). By induction and the fact that $\mathcal{F}_{n+1} \mathcal{F}_n^{-1}$ induces an exponential trichotomy of the form (2.22)–(2.24) (again thanks to Proposition 9.4) we have the bounds for all $\xi \in X$:

$$(9.24) \quad \|\mathcal{G}_n Q^- \xi\|_X \leq C e^{-\gamma(n-m)} \|\mathcal{G}_m Q^- \xi\|_X, \quad n \geq m \geq 0,$$

$$(9.25) \quad \|\mathcal{G}_m Q^+ \xi\|_X \leq C e^{-\gamma(n-m)} \|\mathcal{G}_n Q^+ \xi\|_X, \quad n \geq m \geq 0,$$

$$(9.26) \quad \|\mathcal{G}_n(1 - Q^- - Q^+) \xi\|_X \leq C e^{\gamma_0 |n-m|} \|\mathcal{G}_m(1 - Q^- - Q^+) \xi\|_X, \quad n, m \geq 0.$$

where $\gamma \approx T_* \alpha$ and $\gamma_0 = O(1)$ uniform in T_* . In addition by an energy estimate such as (2.4), we have the uniform growth rate $\|\mathcal{G}_n \mathcal{G}_m^{-1}\|_{\mathcal{B}(X)} \leq C e^{C|n-m|}$. A simple argument then shows that the final estimate:

$$\|\mathcal{G}_n Q^\pm \mathcal{G}_n^{-1}\|_{\mathcal{B}(X)} \lesssim 1,$$

follows automatically from (9.24)–(9.26) (see Chapter 2 of [5]). Thus we have (9.17)–(9.19) which suffices for our application. It remains to prove Proposition 9.7.

Proof of Proposition 9.7. By instead considering $\tilde{\mathcal{G}}_n = e^{\frac{1}{2}(\gamma+\gamma_0)n} \mathcal{G}_n$ and $\tilde{\mathcal{F}}_n = e^{\frac{1}{2}(\gamma+\gamma_0)n} \mathcal{F}_n$ to construct P^- , and similarly $\tilde{\mathcal{G}}_n = e^{-\frac{1}{2}(\gamma+\gamma_0)n} \mathcal{G}_n$ and $\tilde{\mathcal{F}}_n = e^{-\frac{1}{2}(\gamma+\gamma_0)n} \mathcal{F}_n$ to construct P_+ , we can reduce to the more standard situation where there exists a single $\gamma > 0$ and a single projection Q with:

$$(9.27) \quad \|\mathcal{G}_n Q \mathcal{G}_m^{-1}\|_{\mathcal{B}(X)} \leq C e^{-\gamma(n-m)}, \quad n \geq m \geq 0,$$

$$(9.28) \quad \|\mathcal{G}_m(1 - Q) \mathcal{G}_n^{-1}\|_{\mathcal{B}(X)} \leq C e^{-\gamma(n-m)}, \quad n \geq m \geq 0,$$

in which case we are trying to construct a single projection P for the \mathcal{F}_n evolution with the bounds:

$$(9.29) \quad \|\mathcal{F}_n P \mathcal{F}_m^{-1}\|_{\mathcal{B}(X)} \leq C' e^{-\gamma'(n-m)}, \quad n \geq m \geq 0,$$

$$(9.30) \quad \|\mathcal{F}_m(1 - P) \mathcal{F}_n^{-1}\|_{\mathcal{B}(X)} \leq C' e^{-\gamma'(n-m)}, \quad n \geq m \geq 0,$$

where $\gamma' = \gamma - C\epsilon$. To both uniquely determine P and insure that the two constructions are compatible, we will further require that $\text{Range}(1 - P) = \text{Range}(1 - Q)$, or equivalently that

$$(9.31) \quad P(1 - Q) = Q(1 - P) = 0.$$

The argument is now essentially that of Chapter 4 in [5]. We postpone that for a moment in order to show how the two steps mix together.

The first step produces the projector P^- , with the additional property that

$$Q^-(1 - P^-) = P^-(Q^+ + Q^0) = 0.$$

The second step produces the projector P^+ , with the additional property that

$$(1 - P^+)Q^+ = (Q^0 + Q^-)P^+ = 0.$$

These two relations imply that

$$P^- P^+ = P^-(Q^+ + Q^0 + Q^-)P^+ = 0$$

On the other hand, matching the decay properties of solutions we see that $\text{Range}(P^-) \subset \text{Range}(1 - P^+)$, i.e. $P^+P^- = 0$. This shows that the projectors P^+ and P^- are commuting.

We now return to the construction of P as in (9.29), (9.30) given Q as in (9.27), (9.28). First, we write the evolution law for \mathcal{G}_n as a discrete non-autonomous ODE. Setting $\Delta\mathcal{G}_n = \mathcal{G}_{n+1} - \mathcal{G}_n$ we have:

$$(9.32) \quad \Delta\mathcal{G}_n = \mathcal{A}(n)\mathcal{G}_n, \quad \text{where } \mathcal{A}(n) = (\mathcal{G}_{n+1}\mathcal{G}_n^{-1} - I).$$

On the other hand one can consider the inhomogeneous equation:

$$(9.33) \quad \Delta\mathcal{F}_n = \mathcal{A}(n)\mathcal{F}_n + \mathcal{K}_n,$$

which can be solved via Duhamel's principle with initial data $M \in \mathcal{B}(X)$ via:

$$\mathcal{F}_n = \mathcal{G}_n M + \sum_{j=0}^n \mathcal{G}_n \mathcal{G}_j^{-1} \mathcal{K}_{j-1}, \quad \text{where } \mathcal{K}_{-1} = 0.$$

Choosing $M = M_0 - \sum_{j=0}^{\infty} (I - Q)\mathcal{G}_j^{-1} \mathcal{K}_{j-1}$ and assuming the sum converges we can also write the previous line as:

$$\mathcal{F}_n = \mathcal{G}_n M_0 + \sum_{j=0}^n \mathcal{G}_n Q \mathcal{G}_j^{-1} \mathcal{K}_{j-1} - \sum_{j=n+1}^{\infty} \mathcal{G}_n (I - Q) \mathcal{G}_j^{-1} \mathcal{K}_{j-1}, \quad \text{where } \mathcal{K}_{-1} = 0.$$

Now the equation for \mathcal{F}_n is of the form (9.33) where $\mathcal{K}_n = \mathcal{B}(n)\mathcal{F}_n$ with $\mathcal{B}(n) = \mathcal{F}_{n+1}\mathcal{F}_n^{-1} - \mathcal{G}_{n+1}\mathcal{G}_n^{-1}$ producing a small perturbation of the dynamics governed by $\mathcal{A}(n)$. To construct the projection P associated to the stable evolution for \mathcal{F}_n we solve the fixed point equation $\mathcal{J}_n = I(\mathcal{J}_n)$ where:

$$I(\mathcal{J}_n) = \mathcal{G}_n Q + \sum_{j=0}^n \mathcal{G}_n Q \mathcal{G}_j^{-1} \mathcal{B}(j-1) \mathcal{J}_{j-1} - \sum_{j=n+1}^{\infty} \mathcal{G}_n (I - Q) \mathcal{G}_j^{-1} \mathcal{B}(j-1) \mathcal{J}_{j-1}, \quad \text{where } \mathcal{B}_{-1} = 0.$$

By the smallness of $\mathcal{B}(n)$ and the estimates (9.27)–(9.28), the map $\mathcal{J}_n \mapsto I(\mathcal{J}_n)$ is a contraction on the space $\|\mathcal{J}_n\| = \sup_{n \geq 0} \|e^{\gamma' n} \mathcal{J}_n\|_{\mathcal{B}(X)}$ and therefore has a unique fixed point which we also denote by \mathcal{J}_n . Since this solves the homogeneous evolution equation for \mathcal{F}_n we have $\mathcal{J}_n = \mathcal{F}_n P$ where $P = \mathcal{J}_0$. It remains to show P is a projection which satisfies the bounds (9.29) and (9.30) as well as the relation (9.31).

For the latter we observe that applying the fixed point formula for $n = 0$ yields $P = Q + (1 - Q)R$ where $\|R\|_{\mathcal{B}(X)} \lesssim \epsilon$. Thus $1 - P = (1 - Q)(1 - R)$, and (9.31) easily follows.

Multiplying the fixed point equation $\mathcal{F}_m P = I(\mathcal{F}_m P)$ by $\mathcal{G}_n Q \mathcal{G}_m^{-1}$ we get the auxiliary formula:

$$(9.34) \quad \mathcal{G}_n Q \mathcal{G}_m^{-1} \mathcal{F}_m P = \mathcal{G}_n Q + \sum_{j=0}^m \mathcal{G}_n Q \mathcal{G}_j^{-1} \mathcal{B}(j-1) \mathcal{F}_{j-1} P.$$

Setting $n = m = 0$ in this last line yields $QP = Q$. This implies that the fixed point formula also yields the identity $\mathcal{F}_m P^2 = I(\mathcal{F}_m P^2)$, so by uniqueness of the fixed point and evaluation at $n = 0$ we must have $P^2 = P$.

Next, we can subtract (9.34) from the fixed point equation $\mathcal{F}_n P = I(\mathcal{F}_n P)$ for $n > m$ which gives:

$$\mathcal{F}_n P = \mathcal{G}_n Q \mathcal{G}_m^{-1} \mathcal{F}_m P + \sum_{j=m+1}^n \mathcal{G}_n Q \mathcal{G}_j^{-1} \mathcal{B}(j-1) \mathcal{F}_{j-1} P - \sum_{j=n+1}^{\infty} \mathcal{G}_n (I - Q) \mathcal{G}_j^{-1} \mathcal{B}(j-1) \mathcal{F}_{j-1} P.$$

Then applying (9.20), (9.27)–(9.28) and some simple estimates (see [5, Chapter 4]), we get for all $\xi \in X$:

$$\|\mathcal{F}_n P \xi\|_X \lesssim e^{-\gamma'(n-m)} \|\mathcal{F}_m P \xi\|_X, \quad n \geq m \geq 0.$$

Finally, by performing algebraic similar to those above we arrive at the identity (again for $n > m$):

$$\begin{aligned} \mathcal{F}_m (I - P) &= \mathcal{G}_m (I - Q) \mathcal{G}_n^{-1} \mathcal{F}_n (I - P) + \sum_{j=0}^m \mathcal{G}_m Q \mathcal{G}_j^{-1} \mathcal{B}(j-1) \mathcal{F}_{j-1} (I - P) \\ &\quad - \sum_{j=m+1}^n \mathcal{G}_m (I - Q) \mathcal{G}_j^{-1} \mathcal{B}(j-1) \mathcal{F}_{j-1} (I - P), \end{aligned}$$

which suffices to estimate:

$$\|\mathcal{F}_m(I - P)\xi\|_X \lesssim e^{-\gamma'(n-m)} \|\mathcal{F}_n(I - P)\xi\|_X, \quad n \geq m \geq 0.$$

□

REFERENCES

- [1] Serge Alinhac. On the Morawetz–Keel–Smith–Sogge inequality for the wave equation on a curved background. *Publ. Res. Inst. Math. Sci.*, 42(3):705–720, 2006.
- [2] János Bognár. *Indefinite inner product spaces*. Springer-Verlag, New York-Heidelberg, 1974. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 78.
- [3] Jean-Marc Bouclet and Nikolay Tzvetkov. On global Strichartz estimates for non-trapping metrics. *J. Funct. Anal.*, 254(6):1661–1682, 2008.
- [4] Nicolas Burq, Fabrice Planchon, John G. Stalker, and A. Shadi Tahvildar-Zadeh. Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay. *Indiana Univ. Math. J.*, 53(6):1665–1680, 2004.
- [5] W. A. Coppel. *Dichotomies in stability theory*. Lecture Notes in Mathematics, Vol. 629. Springer-Verlag, Berlin-New York, 1978.
- [6] Shin-ichi Doi. Remarks on the Cauchy problem for Schrödinger-type equations. *Comm. Partial Differential Equations*, 21(1-2):163–178, 1996.
- [7] Vladimir Georgescu, Christian Gérard, and Dietrich Häfner. Resolvent and propagation estimates for Klein-Gordon equations with non-positive energy. *J. Spectr. Theory*, 5(1):113–192, 2015.
- [8] Christian Gérard. Scattering theory for Klein-Gordon equations with non-positive energy. *Ann. Henri Poincaré*, 13(4):883–941, 2012.
- [9] Kunio Hidano and Kazuyoshi Yokoyama. A remark on the almost global existence theorems of Keel, Smith and Sogge. *Funkcial. Ekvac.*, 48(1):1–34, 2005.
- [10] Markus Keel, Hart F. Smith, and Christopher D. Sogge. Almost global existence for some semilinear wave equations. *J. Anal. Math.*, 87:265–279, 2002. Dedicated to the memory of Thomas H. Wolff.
- [11] Carlos E. Kenig, Gustavo Ponce, and Luis Vega. On the Zakharov and Zakharov-Schulman systems. *J. Funct. Anal.*, 127(1):204–234, 1995.
- [12] Herbert Koch and Daniel Tataru. Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients. *Comm. Pure Appl. Math.*, 54(3):339–360, 2001.
- [13] Herbert Koch and Daniel Tataru. Carleman estimates and absence of embedded eigenvalues. *Comm. Math. Phys.*, 267(2):419–449, 2006.
- [14] Heinz Langer. Spectral functions of definitizable operators in Krein spaces. In *Functional analysis (Dubrovnik, 1981)*, volume 948 of *Lecture Notes in Math.*, pages 1–46. Springer, Berlin-New York, 1982.
- [15] Heinz Langer, Branko Najman, and Christiane Tretter. Spectral theory of the Klein-Gordon equation in Pontryagin spaces. *Comm. Math. Phys.*, 267(1):159–180, 2006.
- [16] Jeremy Marzuola, Jason Metcalfe, and Daniel Tataru. Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations. *J. Funct. Anal.*, 255(6):1497–1553, 2008.
- [17] Jason Metcalfe and Christopher D. Sogge. Long-time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods. *SIAM J. Math. Anal.*, 38(1):188–209, 2006.
- [18] Jason Metcalfe and Christopher D. Sogge. Global existence of null-form wave equations in exterior domains. *Math. Z.*, 256(3):521–549, 2007.
- [19] Jason Metcalfe and Daniel Tataru. Global parametrices and dispersive estimates for variable coefficient wave equations. *Math. Ann.*, 353(4):1183–1237, 2012.
- [20] Jason Metcalfe, Daniel Tataru, and Mihai Tohaneanu. Price’s law on nonstationary space-times. *Adv. Math.*, 230(3):995–1028, 2012.
- [21] Cathleen S. Morawetz. Exponential decay of solutions of the wave equation. *Comm. Pure Appl. Math.*, 19:439–444, 1966.
- [22] Cathleen S. Morawetz. Time decay for the nonlinear Klein-Gordon equations. *Proc. Roy. Soc. Ser. A*, 306:291–296, 1968.
- [23] Cathleen S. Morawetz. Decay for solutions of the exterior problem for the wave equation. *Comm. Pure Appl. Math.*, 28:229–264, 1975.
- [24] Cathleen S. Morawetz, James V. Ralston, and Walter A. Strauss. Decay of solutions of the wave equation outside nontrapping obstacles. *Comm. Pure Appl. Math.*, 30(4):447–508, 1977.
- [25] Hart F. Smith and Christopher D. Sogge. Global Strichartz estimates for nontrapping perturbations of the Laplacian. *Comm. Partial Differential Equations*, 25(11-12):2171–2183, 2000.
- [26] Gigliola Staffilani and Daniel Tataru. Strichartz estimates for a Schrödinger operator with nonsmooth coefficients. *Comm. Partial Differential Equations*, 27(7-8):1337–1372, 2002.
- [27] Jacob Sterbenz. Angular regularity and Strichartz estimates for the wave equation. *Int. Math. Res. Not.*, (4):187–231, 2005. With an appendix by Igor Rodnianski.
- [28] Walter A. Strauss. Dispersal of waves vanishing on the boundary of an exterior domain. *Comm. Pure Appl. Math.*, 28:265–278, 1975.
- [29] Daniel Tataru. Local decay of waves on asymptotically flat stationary space-times. *Amer. J. Math.*, 135(2):361–401, 2013.